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On the Mechanical Foundation of the Theory of Special Relativity.

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Summary. — From the given system of five axioms expressing the mechanical aspect of physical reality there follow the well-known expressions for the mass and the energy of material particles, the existence of the limit-velocity for them and its independency from the relative velocity of the galilean systems of reference, further Einstein's law of the addition of velocities and the Lorentz-Einstein transformation. Finally it is shown briefly that Newton's « tempus absolutum verum et mathematicum » is mutually conditioned with the constant mass of material particles and vice versa.

Introduction.

It is well known that for material particles moving with great velocities the laws of Classical Mechanics are not valid but those of the Theory of Special Relativity. In his famous treatise on the fundamental equations for electromagnetic processes in the moving media H. MINKOWSKI (1) expressed the opinion that a reform of Mechanics would be desirable where instead of Newton's Principle of Relativity with $c = \infty$ a new one should be introduced with a *finite* limit-velocity, representing the essential point for the axiomatic foundation of Mechanics.

(*) At present in München.

(1) H. MINKOWSKI: *Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Medien* (*Gött. Nachr.*, 1908), p. 53.

Many attempts have been made (2)—mostly of mathematical character—to derive this theory *independently* of Electrodynamics. In this paper the laws of Relativistic Mechanics will be derived from a system of five axioms of a purely mechanical character.

1. – The given system of axioms.

Ax. 1. – For the transformation of co-ordinates and the time between two galilean systems of reference S and S' and their relative velocities there are relations

$$x' = a_{11}x + a_{12}t \,, \tag{1}$$

$$t' = a_{21}x + a_{22}t \,, \tag{2}$$

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = v \qquad \text{when} \qquad \dot{x}' = \frac{\mathrm{d}x'}{\mathrm{d}t'} = 0 \,, \tag{3}$$

$$\dot{x}' = \frac{\mathrm{d}x'}{\mathrm{d}t'} = -v \qquad \text{when} \qquad \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = 0 \,, \tag{4}$$

where v represents the constant relative velocity of the systems S and S' and a_{11}, a_{12}, a_{21}, a_{22} are the coefficients to be determined.

Ax. 2. – The mass and the energy of every material particle are always finite and between them there exists the relation

$$E = \varkappa m \,, \tag{5}$$

$\varkappa$ being a constant, $m = m(|\dot{\mathfrak{r}}|)$ their mass which could be changed if the absolute value $|\dot{\mathfrak{r}}|$ of the velocity of material particles would change, while E represents their energy (*).

Ax. 3. – For the movement of material particles Newton's law is valid

$$\mathfrak{F} = \frac{\mathrm{d}m(|\dot{\mathfrak{r}}|)\dot{\mathfrak{r}}}{\mathrm{d}t} \,. \tag{6}$$

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(2) PH. FRANK and H. ROTHE: *Ann. d. Phys.*, **34**, 825 (1911); *Phys. Zeits.*, **13**, 750 (1912); G. N. LEWIS and C. TOLMAN: *Phil. Mag.*, **18**, 510 (1909); Y. UENO: *Progr. Theor. Phys.*, **9**, 74 (1953); H. TAKENO: *Progr. Theor. Phys.*, **12**, 129 (1954).

(*) The proportionality between the mass and the energy was discovered empirically, before the theory of relativity, by Hasenörl and Lebedev, in connection with the electromagnetic radiation.

Ax. 4. – In all galilean systems of reference the physical laws have the same analytical form (The Principle of Special Relativity) .

Ax. 5. – Between the x— and x'— component of the force $\mathfrak{F}$ acting on a material particle in direction of the axes X and X' in the galilean systems of reference S and S' respectively, the relation

$$\frac{\mathrm{d}m(\dot{x})\dot{x}}{\mathrm{d}t} = \frac{\mathrm{d}m(\dot{x}')\dot{x}'}{\mathrm{d}t'} \tag{7}$$

is valid.

2. – The consequences resulting from the given system of axioms.

Let us take first that the material particle is resting in the system S'. Then its velocity in S and S' is $\dot{x} = v$ and $\dot{x}' = 0$ respectively. Suppose that the force $\mathfrak{F}$ acting in the direction of the X axis in the time interval $\mathrm{d}t$ on the material particle of the mass m covering, with regard to the system S on the X-axis, the way $\mathrm{d}x = \mathrm{d}s$ with the velocity v has changed the value of its energy by $\mathrm{d}E$. Then considering (5) we have

$$\mathrm{d}E = \varkappa\,\mathrm{d}m \tag{8}$$

and for the corresponding work measured in the system S

$$\mathrm{d}E = F\cdot\mathrm{d}s\,. \tag{9}$$

From (6), (8) and (9) it follows

$$\frac{\mathrm{d}m}{m} = -\frac{1}{2}\frac{\mathrm{d}(1-(v^2/\varkappa)}{1-(v^2/\varkappa)}\,. \tag{10}$$

Putting

$$\varkappa = C^2 \tag{11}$$

in (10) we obtain by integration

$$m = \frac{m_0}{\sqrt{1-(v^2/C^2)}}\,, \tag{12}$$

as the well-known expression for the mass of a moving particle, m_0 being its rest mass.

With the aid of (5) and the equation (6) in its scalar form and (12) follows the expression for the energy of the material particle and the force acting on it

$$E = \frac{m_0C^2}{\sqrt{1-(v^2/C^2)}}\,, \tag{13}$$

and

$$F = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_0 v}{\sqrt{1-(v^2/C^2)}}\right), \tag{14}$$

respectively.

According to Ax. 2, the mass and the energy of the material particle being always finite it follows from (12) and (13) that it can never be $v = C$.

Therefrom we conclude:

For the velocities of material particles there exists *an upper limit* called *the limit-velocity* which material particles can never attain.

Let us take now that a material particle is moving relatively to the galilean systems of reference S and S' with the velocity

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}, \tag{15}$$

and

$$\dot{x}' = \frac{\mathrm{d}x'}{\mathrm{d}t'}, \tag{16}$$

respectively, the constant relative velocity of the system S' measured from S being v.

Considering Ax. 4 and the formula (13), the change of the energy of the given particle caused by the acting force and the corresponding work in the system S and S' shall be

$$\mathrm{d}E = \frac{m_0 \ddot{x}\,\mathrm{d}x}{(1-(\dot{x}^2/C^2))^{\frac{3}{2}}}, \tag{17}$$

$$\mathrm{d}E = F\,\mathrm{d}x, \tag{18}$$

and

$$\mathrm{d}E' = \frac{m_0 \ddot{x}'\,\mathrm{d}x'}{(1-(\dot{x}'^2/C'^2))^{\frac{3}{2}}}, \tag{19}$$

$$\mathrm{d}E' = F'\,\mathrm{d}x', \tag{20}$$

where C and C' are two constants for which we do not know *a priori* if they are equal.

From (1), (7), (18), (20) and the relation

$$\mathrm{d}t = \frac{\mathrm{d}x}{\dot{x}}, \tag{21}$$

we obtain

$$\mathrm{d}E' = \mathrm{d}E \cdot \left(a_{11} + \frac{a_{12}}{\dot{x}}\right). \tag{22}$$

By substitution of (17) and (19) in (22), considering that

$$\mathrm{d}x' = \left(a_{11} + \frac{a_{12}}{\dot{x}}\right) \mathrm{d}x\,, \tag{23}$$

it results the relation

$$\frac{\ddot{x}}{(1-(\dot{x}^2/C^2))^{\frac{3}{2}}} = \frac{\ddot{x}'}{(1-(\dot{x}'^2/C'^2))^{\frac{3}{2}}}\,, \tag{24}$$

where from the coefficients a_{11}, a_{12}, a_{21}, a_{22} of the linear transformations (1), (2) can be determined.

With the aid of (24) and the relations

$$\dot{x}' = \frac{a_{11}\dot{x} + a_{12}}{a_{21}\dot{x} + a_{22}}\,, \tag{25}$$

$$\ddot{x}' = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{21}\dot{x} + a_{22})^3} \cdot \ddot{x}\,, \tag{26}$$

which follow from (1) and (2), we get

$$\dot{x}^2 \cdot \left[\left[a_{21}^2 C'^2 - a_{11}^2 + \frac{C'^2}{C^2}(a_{11}a_{22} - a_{12}a_{21})^{\frac{2}{3}}\right] + 2\dot{x} \cdot (a_{21}a_{22}C'^2 - a_{11}a_{12}) + \right. \tag{27}$$

$$\left. + a_{22}^2 \cdot C'^2 - a_{12}^2 - \frac{C'^2}{C^2}(a_{11}a_{22} - a_{12}a_{21})^{\frac{2}{3}} \cdot C^2 = 0\,,\right.$$

wherefrom, owing to the changeability of $\dot{x}$, results

$$a_{21}^2 C'^2 - a_{11}^2 + \frac{C'^2}{C^2}(a_{11}a_{22} - a_{12}a_{21})^{\frac{2}{3}} = 0\,, \tag{28}$$

$$a_{21}a_{22}C'^2 - a_{11}a_{12} = 0\,, \tag{29}$$

$$a_{22}^2 C'^2 - a_{12}^2 - \frac{C'^2}{C^2} \cdot (a_{11}a_{22} - a_{12}a_{21})^{\frac{2}{3}} \cdot C^2 = 0\,. \tag{30}$$

Putting

$$\frac{C'^2}{C^2}(a_{11}a_{22} - a_{12}a_{21})^{\frac{2}{3}} = \alpha\,, \tag{31}$$

and dividing (28) by a_{11}^2 follows

$$\left(\frac{a_{21}}{a_{11}}\,C'\right)^2 - 1 + \frac{\alpha}{a_{11}^2} = 0 \tag{32}$$

and also instead of (30)

$$a_{22}^2 C'^2 - a_{12}^2 - \alpha C'^2 = 0\,. \tag{33}$$

Considering Ax. 1 and (25) we have

$$\dot{x} = v = -\frac{a_{12}}{a_{11}}\,, \tag{34}$$

$$\dot{x}' = -v = \frac{a_{11}}{a_{22}} \tag{36}$$

and therefrom

$$a_{11} = a_{22}\,, \tag{36}$$

From (29), (34) and (36) we get

$$\frac{a_{21}}{a_{11}}\,C' = -\frac{v}{C'}\,, \tag{37}$$

while from (32) and (37) results

$$\frac{\alpha}{a_{11}^2} = 1 - \frac{v^2}{C'^2}\,. \tag{38}$$

Finally from (33), (34), (36) and (38) follows

$$C'^2 - C^2 - v^2 \cdot \left(1 - \frac{C^2}{C'^2}\right) = 0 \tag{39}$$

and herefrom, owing to the changeability of the constant relative velocity v of galilean systems of reference S and S', the fundamental relation

$$C = C'\,. \tag{40}$$

This means: *The limit-velocity of material particles is independent from the relative velocity of galilean systems of reference S and S' and represents, consequently, a universal constant.*

This is *the principle of the constancy of the limit-velocity of material particles.*

Experiments show that the light-velocity c *in vacuo* represents the limit-velocity for material particles. For that reason we can identify the limit-

velocity C of material particles with the light-velocity c *in vacuo*,

$$C = c . \tag{41}$$

From the relations (31), (35), (36), (37), (38) and (40) there follows the coefficients of the Lorentz-Einstein transformation and, considering (25), Einstein's law of the addition of velocities for material particles ([3]).

In the case of Newton's classical supposition of the *temporis absoluti* from Ax. 1, Ax. 5 and (36) ([4]) it follows that the mass of material particles must be constant. Oppositely, supposing that the mass of material particles is constant, from Ax. 1, Ax. 5 and (36), due to the changeability of $\dot{x}$, we get the Galilei-Newton transformation. Consequently, we conclude, that Newton's « *tempus absolutum verum et mathematicum* » is mutually conditioned by the constant mass of material particles and vice versa ([5]).

([3]) A. EINSTEIN: *The Meaning of Relativity* (Princeton, 1953).

([4]) I. NEWTON: *Philosophiae naturalis principia mathematica* (London, 1687).

([5]) K. STIEGLER: *Acad. R. de Belgique*, Bulletin (Cl. Sci.) T. 39 (Bruxelles, 1953), p. 1052.

RIASSUNTO (*)

Da un dato sistema di cinque assiomi che esprimono l'aspetto meccanico della realtà fisica segue la nota espressione per la massa e l'energia delle particelle materiali, l'esistenza di una velocità limite per le particelle e la sua indipendenza dalla velocità relativa dei sistemi di riferimento galileani; inoltre la legge di Einstein per la somma delle velocità e la trasformazione di Lorentz-Einstein. Si dimostra infine brevemente, come il « tempus absolutus verus et mathematicus » dipende dalla massa costante delle particelle materiali e viceversa.

(*) *Traduzione a cura della redazione.*

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Einstein's Field Equations.

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Summary. — The nature of a static line element in the relativity theory is investigated and it is shown that both the force and the torque on an isolated body must vanish if the field equations for empty space are to have a static solution in its neighbourhood. It is also shown how a general solution for a static electro-gravitational field can be constructed by a method of successive approximation.

1. – Introduction.

Weyl showed that a static metric possessing rotational symmetry can be represented by a line element of the form

$$\mathrm{d}s^2 = -e^{-W}[e^{V}(\mathrm{d}z^2 + \mathrm{d}r^2) + r^2\,\mathrm{d}\theta^2] + e^{W}\mathrm{d}t^2 .$$

When this line element is used, the field equations for empty space reduce to

$$\Delta W = 0 , \tag{1}$$

$$V_1 = rW_1W_2 , \qquad V_2 = \tfrac{1}{2}r(W_2^2 - W_1^2) , \tag{2}$$

where, Δ is the Laplacian operator in cylindrical co-ordinates, and the subscripts 1 and 2 denote differentiation with respect to z and r respectively. After Eq. (1) has been solved by introducing an axially symmetric density distribution the function V can be determined in terms of W by evaluating the integral

$$I = \int (V_1\,\mathrm{d}z + V_2\,\mathrm{d}r) .$$

If V is to be single-valued this integral must vanish on all closed curves lying entirely in empty space. In a previous paper [1] (to be referred to as I) it was shown by very simple arguments that the condition is fulfilled only when the gravitational force on every isolated body vanishes. The object of the present paper is to generalize this result to the case of an arbitrary static field. The difficulty in the generalization has been pointed out by ROSEN [2] whose method consists in expanding the field in the neighbourhood of the particle in powers of $(r - m/2)$ and then calculating the stress on a sphere of radius r as $r \to m/2$. It will be shown here that the result can be deduced more simply from the theorem that in any Riemannian space the co-variant divergence of the tensor $T^{\nu}_{\mu} = G^{\nu}_{\mu} - \frac{1}{2}\delta^{\nu}_{\mu} G$ vanishes identically. Although the physical content of the result is independent of the choice of co-ordinates, discussions are considerably simplified by using a special type of space co-ordinates. In the fundamental form $ds^2 = -(\delta_{ab} + h_{ab})\,dx^a dx^b + e^W dt^2$ the coefficients h_{ab} are usually of the same order of magnitude as W. Consequently, in the integral $\iiint T^{\nu}_{\mu\nu}\,dv$ the second order terms we are looking for get inextricably mixed up with numerous other terms which depend on the type of co-ordinates used and have no particular physical significance. The possibility of choosing a co-ordinate system in which no such undesirable terms will occur arises from a remarkable property of static fields revealed by the exact solution of I, Section **4**. It has been mentioned in I, Section **5** that the solution gives a field in which the gravitational and electric forces balance at every point. For this to happen every piece of matter must carry a disproportionately small amount of electric charge. Physically this means that the deformation of the metric caused by the electric field will be very small compared with that caused by the presence of matter. The complete withdrawal of the electric field will therefore leave a metric with a line element differing only slightly from

$$ds^2 = -e^{-W}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + e^W dt^2 . \tag{3}$$

In other words, any static space compatible with the field equations $G_{\mu\nu} = 0$ is to a good approximation conformal to a flat space. We can, therefore, safely assume that a class of co-ordinates always exists for which the line element of a static field takes the form

$$ds^2 = -e^{-W}\bar{g}_{ab}\,dx^a dx^b + e^W dt^2 , \tag{4}$$

where the coefficients $\bar{g}_{ab}$ differ from Cartesian values by small quantities of the second order. When such a co-ordinate system is used all unwanted terms

(1) S. D. MAJUMDAR: *Phys. Rev.*, **72**, 390 (1947).
(2) N. ROSEN: *Rev. Mod. Phys.*, **21**, 503 (1949).

drop out from the equation $\iiint T^{\nu}_{\mu\nu}\,dv = 0$ and the theorem is established. The line element retains this character even in the presence of an electric field if the electrostatic potential φ is of the same order of magnitude as W. In such cases the dynamical effects of the two fields are also of the same order. We will, therefore, consider a combined electro-gravitational field of a perfectly general type subject only to the above qualitative restriction.

The above clarification leads to the possibility of constructing a general solution of Einstein's equations for an electro-gravitational field by a method of successive approximation. Although it is possible to obtain a formal solution up to any desired order, the validity of the iterative process is questionable and the results are given explicitly only up to the second order.

2. – The fundamental equations.

As in I, Section **4**, we write the line element of the static field in the form (4). (In all formulae Latin indices will take the values 1, 2, 3 and Greek indices the values 1, 2, 3, 4). With this form of the line element the components of the tensor $T^{\nu}_{\mu} = G^{\nu}_{\mu} - \frac{1}{2}\delta^{\nu}_{\mu}G$ are, by I, Eqs. (25) and (29)

$$-\dot{T}^{j}_{i} = -T^{j}_{i}\sqrt{-g} = \dot{\overline{T}}^{j}_{i} + \sqrt{\overline{g}}\,(\tfrac{1}{2}W_iW^j - \tfrac{1}{4}\delta^{j}_{i}\Delta_1 W)$$

$$-e^{-W}T^{4}_{4} = \Delta_2 W - \tfrac{1}{2}\overline{G} - \tfrac{1}{4}\Delta_1 W\,, \qquad -e^{-W}G^{4}_{4} = \tfrac{1}{2}\Delta_2 W\,, \qquad T^{4}_{4} = 2G^{4}_{4} + T^{a}_{a}\,,$$

where,

$$W^j = \overline{g}^{ja}W_a\,, \qquad \Delta_1 W = \overline{g}^{ab}W_aW_b\,, \qquad \Delta_2 W = \overline{g}^{ab}W_{;ab}$$

and $W_{;ab}$ is the second covariant derivative of W in the space defined by the coefficients $\overline{g}_{ij}$. The components of the electromagnetic energy tensor are, by I, Eqs. (15) and (16)

$$-8\pi E^{j}_{i} = -2\varphi_i\varphi^j + \delta^{j}_{i}\Delta_1\varphi\,, \qquad 8\pi E^{4}_{4} = \Delta_1\varphi\,.$$

The field equations for empty space are, therefore,

$$\overline{T}^{j}_{i} + \tfrac{1}{2}W_iW^j - \tfrac{1}{4}\delta^{j}_{i}\Delta_1 W = 2e^{-W}\varphi_i\varphi^j - e^{-W}\delta^{j}_{i}\Delta_1\varphi\,, \tag{5}$$

$$\Delta_2 W - \tfrac{1}{2}\overline{G} - \tfrac{1}{4}\Delta_1 W = e^{-W}\Delta_1\varphi\,, \quad \text{or} \quad \Delta_2 W = 2e^{-W}\Delta_1\varphi\,. \tag{6}$$

In the regions occupied by matter

$$T^{\nu}_{\mu} = -8\pi\,(M^{\nu}_{\mu} + E^{\nu}_{\mu})\,,$$

where M_{μ}^{ν} is the material energy tensor. The potential φ satisfies the equation

$$\Delta_2 \varphi \triangleq W_a \varphi^a - 4\pi\sigma \,, \tag{7}$$

where σ is the charge density in relativistic electrostatic units. The non-vanishing components of the electromagnetic field tensor are

$$2\sqrt{\pi} F_{4i} = \varphi_i \,, \qquad 2\sqrt{\pi} F^{4i} = -\varphi^i \,.$$

Finally, in terms of the tensor density the identity $T_{\mu\nu}^{\nu} = 0$ becomes

$$(\partial \dot{T}_{\mu}^{\beta} / \partial x^{\beta}) = \{\mu\beta, \alpha\} \dot{T}_{\alpha}^{\beta} \,.$$

The Eqs. (5)–(7) are the fundamental equations of the electrogravitational field. The structure of these equations provides the mathematical basis for the assumption that the $\overline{g}_{ij}$ differ from Euclidean values by small quantities of the second order. If the second order terms are all neglected, we have $\overline{T}_i^j = 0$, or $\overline{G}_{ij} = 0$. Since the vanishing of the contracted Riemann tensor is a necessary and sufficient condition for the flatness of a three-dimensional space, it follows that to a first approximation a solution of the equations is given by $\overline{g}_{ij} = \delta_{ij}$. It is, therefore, clear that, if the neglected terms are taken into account, $\overline{g}_{ij}$ will differ from δ_{ij} only by quantities of the second order so long as we are dealing with matter of low density confined to a small region of space.

3. – Proof of the theorem in the general case.

The theorem to be proved may be stated as follows:

«If the equations $G_{\mu\nu} = 0$ are to have a consistent solution in the space surrounding an isolated body, the resultant force and the resultant torque on the body due to the field must vanish ».

For an unambiguous formulation of the theorem it is necessary to express these conditions in terms of integrals over a surface S enclosing the isolated body and lying entirely in empty space. This is possible, but the physical meaning of the conditions becomes somewhat obscure when they are so expressed.

The first part of the theorem is proved immediately by integrating the equation

$$-8\pi(\partial \dot{M}_i^{\alpha} / \partial x^{\alpha}) - 8\pi \dot{E}_{i\alpha}^{\alpha} = \{i\beta, \alpha\}(\dot{T}_{\alpha}^{\beta} + 8\pi \dot{E}_{\alpha}^{\beta})$$

over the volume enclosed by S and neglecting all terms of the third and higher orders. The result is

$$0 \simeq \iiint (8\pi \dot{E}_{i\alpha}^{\nu} + \{i, 44\}\dot{T}_4^4)\,\mathrm{d}v = \iiint \left(8\pi \dot{F}_a^{4a} F_{i4} + \frac{1}{2} W_i \dot{T}_4^4\right) \mathrm{d}v$$

$$\simeq 2\iiint (\varphi_i \nabla^2 \varphi - \psi_i \nabla^2 \psi)\,\mathrm{d}v \simeq 8\pi \iiint (-\sigma\varphi_i + \varrho\psi_i)\,\mathrm{d}v\,,$$

where $\psi = -W/2$ is the gravitational potential of the Newtonian theory. To prove the second part we write

$$x^i(\partial \dot{T}_j^\alpha/\partial x^\alpha) - x^j(\partial \dot{T}_i^\alpha/\partial x^\alpha) = (x^i\{j\beta, \alpha\} - x^j\{i\beta, \alpha\})\,\dot{T}_\alpha^\beta\,.$$

The integration now yields

$$0 \simeq \iiint [(x^i\varphi_j - x^j\varphi_i)\nabla^2\varphi - (x^i\psi_j - x^j\psi_i)\nabla^2\psi]\,\mathrm{d}v\,.$$

This means that the torque acting on the body vanishes. The corresponding surface integral conditions are

$$\iint l_k[\varphi^2(\log\varphi)_{ik} - \psi^2(\log\psi)_{ik}]\,\mathrm{d}S \simeq 0\,, \tag{8}$$

$$\iint l_k[\varphi^2\{(x^i \log\varphi)_{jk} - (x^j \log\varphi)_{ik}\} - \psi^2\{(x^i \log\psi)_{jk} - (x^j \log\psi)_{ik}\}]\,\mathrm{d}S \simeq 0,$$

where the l_k are the direction cosines of the surface normal. With the help of the pseudo-tensor of the gravitational field it seems possible to formulate the exact surface integral conditions corresponding to (8) above. But, in the axially symmetric case it does not lead to any fruitful result.

When the field possesses axial symmetry five of these conditions are identically satisfied. To see what form the remaining one assumes we integrate the exact equation

$$-8\pi(\partial \dot{M}_1^\alpha/\partial x^\alpha) = \{1\beta, \alpha\}\,\dot{T}_\alpha^\beta\,,$$

over an area in the r-z plane obtaining

$$0 = -\frac{1}{2}\iint r W_1 \Delta W\,\mathrm{d}S = -\frac{1}{4\pi}\iiint W_1 \Delta W\,\mathrm{d}v\,.$$

The formulation remains valid even when the sources of the field occur in the form of point or line singularities and, therefore, contains one of Rosen's results as a special case.

Following this line of argument we can also show that, in the static case, the geodesic postulate for the motion of a test particle is a consequence of the field equations for empty space. A picture of a test particle is obtained by assuming T_μ^ν to be exactly zero outside an infinitesimal region where it has vanishingly small values. Introducing locally Cartesian co-ordinates at a point in this region we see that in the limiting case the vanishing of the volume integral leads to the condition $\partial g_{44}/\partial x^i = 0$. This means that the track of a particle at rest is a geodesic in the background metric.

The above considerations can be extended to the case of several bodies widely separated in space and moving slowly under the influence of one another. Some idea of the nature of the coefficients $g_{\mu\nu}$ in this quasi-static case can be obtained by applying a Lorentz transformation to the line element (4). Since the square of the velocity is roughly of the order of W, the transformation provides a logical basis for writing the line element in the form

$$ds^2 = -e^{-W}\,\overline{g}_{ab}\,dx^a\,dx^b + 2g_{i4}\,dx^i\,dt + e^W\,dt^2\,,$$

where, $\overline{g}_{ab} = \delta_{ab} + O(W^2)$ as in the static case, and $g_{i4} = O(W^{\frac{3}{2}})$. Further, it must be assumed that differentiation with respect to time raises the order of smallness of a field quantity by $\frac{1}{2}$. On these assumptions the classical equations of motion of a body in a gravitational field can be easily derived from the identity $\iiint T_{\mu\nu}^\nu\,dv = 0$ if terms only up to the second order are retained. The relativistic effects first become apparent when one goes beyond the second order. The derivation depends solely on the field equations for empty space, $T_\mu^\nu = 0$, and not at all on the connection between the tensor T_μ^ν and the energy momentum tensor. In fact, any assumption regarding the constitution of matter can be avoided by transforming the volume integrals into integrals over a surface lying entirely in empty space [3].

4. – A general solution of the field equations.

An essential step in constructing a solution of the field equations is to restrict as far as possible the freedom of choice of space co-ordinates. Since co-ordinates can be transformed arbitrarily, it appears that we have at our disposal three functions which can be suitably chosen to make g_{ij} vanish for $i \neq j$. As this argument is not at all convincing, we simply assume that a

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[3] A. EINSTEIN, L. INFELD and B. HOFFMANN: *Ann. Math.*, **39**, 65 (1938).

wide class of static fields can be represented by a line element of the form

$$ds^2 = -e^{-W}[e^P(dx^1)^2 + e^Q(dx^2)^2 + e^R(dx^3)^2] + e^W dt^2$$

and then examine the consequences of such an assumption. For such a line element the number of functions to be determined is less than the number of differential equations. That they do not form an over-determined system in the general case under consideration can be demonstrated, at each stage of the approximation, with the help of the identity $T^{\nu}_{\mu\nu} = 0$.

To a first approximation the field equations reduce to

$$\nabla^2 W = 0\,, \qquad \nabla^2 \varphi = 0$$

in agreement with the classical theory. Next, when all terms up to the second order are retained, we have

$$\nabla^2 W = 2\varphi_a \varphi_a \tag{9}$$

$$\nabla^2 \varphi = W_a \varphi_a\,, \tag{10}$$

$$R_{12} = -W_1 W_2 + 4\varphi_1 \varphi_2\,; \quad \text{etc.} \tag{11}$$

$$Q_{33} + R_{22} = W_1^2 - \tfrac{1}{2} W_a W_a - 4\varphi_1^2 + 2\varphi_a \varphi_a\,, \quad \text{etc.} \tag{12}$$

Eq. (9) shows that W remains a harmonic function up to the second order if no electric field is present. It is interesting to note that for $\varphi = \pm W/2$ the right hand sides of Eqs. (11) and (12) vanish, and for $1 \pm \varphi = \exp[W/2]$ the Eqs. (9) and (10) reduce to the same equation $\nabla^2 W = \frac{1}{2} W_a W_a$. Therefore, if W and φ are assumed to be connected by a relation of the form $1 \pm \varphi = \exp[W/2]$, then a special solution of the Eqs. (9)–(12) is obtained by setting $P = Q = R = 0$. But, as shown in I, Section **4**, this is an exact solution of the field equations representing a special type of distribution of matter and charge. For perfectly arbitrary distributions an approximate solution correct to the second order can be obtained by substituting the harmonic functions $W^{(1)}$ and $\varphi^{(1)}$ in the right hand sides of the above equations. From Eqs. (11) the functions P, Q, R are immediately obtained as double integrals:

$$R^{(2)} = \int_{-\infty}^{y} \int_{-\infty}^{x} (-W_1^{(1)} W_2^{(1)} + 4\varphi_1^{(1)} \varphi_2^{(1)})\, dx\, dy\,. \tag{13}$$

Since any metric must approach that of the special relativity theory at great distances from matter and charge, R should tend to zero as x or y tends to

infinity. This condition is not likely to be fulfilled unless the field has reflection symmetry with regard to each one of the three co-ordinate planes. A closer examination of the situation, therefore, forces us to impose this restriction on the solutions that can be obtained by the present method. Further, it will be assumed that the stress components M_i^j for $i \neq j$ vanish even in the regions where the density of matter is different from zero. But this does not in any way restrict the generality of the solution for empty space in which one is primarily interested.

To show that the functions P, Q, R thus obtained also satisfy the Eqs. (12) we make use of the identity

$$T_{i\alpha}^{\alpha} + 8\pi E_{i\alpha}^{\alpha} = 8\pi E_{i\alpha}^{a} .$$

Since $T_i^{\alpha} + 8\pi E_i^{\alpha} = 0$ for $i \neq \alpha$, this can be written as

$$\partial(\dot{T}_i^i + 8\pi \dot{E}_i^i)/\partial x^i = \sum_{\alpha} \{i\alpha, \alpha\}(\dot{T}_{\alpha}^{\alpha} + 8\pi \dot{E}_{\alpha}^{\alpha}) + 8\pi \dot{F}_a^{4a} F_{i4} ,$$

whence,

$$(14) \quad \dot{T}_i^i + 8\pi \dot{E}_i^i = \int [\sum_{\alpha} \{i\alpha, \alpha\}(\dot{T}_{\alpha}^{\alpha} + 8\pi \dot{E}_{\alpha}^{\alpha}) - 8\pi \dot{\sigma} \varphi_i] \mathrm{d}x^i , \quad \text{(not summed over } i\text{)}.$$

In the second approximation this gives

$$\dot{T}_i^i + 8\pi \dot{E}_i^i = -\int \left[\frac{1}{2} W_i \nabla^2 W + 8\pi \sigma \varphi_i\right] \mathrm{d}x^i = 0 .$$

The result can also be established directly by using the Eqs. (13).

Following this procedure we can formally obtain a solution up to any desired order, although the question of convergence remains undecided. The method consists in solving the equations (5) for $i \neq j$ together with the equations (6) and (7) by successive approximation and in showing that the remaining three equations $T_i^i = -8\pi E_i^i$ are also satisfied. The equations to be solved are of the type

$$\nabla^2 W = O(W^2) + f(x^2, y^2, z^2) , \qquad \nabla^2 \varphi = O(W^2) + h(x^2, y^2, z^2) ,$$

$$R_{12} = \tfrac{1}{2}(-R_1 R_2 + R_1 P_2 + R_2 Q_1) - W_1 W_2 + 4e^{-W} \varphi_1 \varphi_2 ,$$

where f and h are arbitrary functions representing the densities of matter and charge. These can be treated as inhomogeneous linear equations by substi-

tuting in the right hand side the functions obtained in the previous approximation. The consistency of the eight equations follows from (14) by induction. Let us suppose that the equations are consistent in the $(n-1)$-th approximation. To get the value of $\dot{T}_i^i+8\pi\dot{E}_i^i$ in the n-th approximation we can substitute in the right hand side of Eq. (14) its value obtained in the previous approximation. But, by hypothesis the latter vanishes at all points unoccupied by matter.

5. – The quasi-static case and the equations of motion.

We conclude this paper by showing how the Newtonian equations of motion of a finite body arise in an elementary way from the field equations for empty space. The problem has been treated in detail by a number of workers ([3-5]). To avoid the phenomenological relation between the tensor T_μ^ν and the energy momentum tensor Einstein, Infeld and Hoffmann represented matter by means of singularities of the field. But all the basic ideas needed for further elaboration of the theory are contained in their paper. Our primary aim is to examine to what extent the motion of a finite body is determined by the field equations for empty space alone. At the outset we, therefore, assume nothing about the interior of matter and simply define it to be a region where the tensor T_μ^ν does not vanish. According to the procedure outlined in Section **3** we then try to derive the equations of linear motion from the identity $\iiint T_{i\nu}^\nu \,\mathrm{d}v = 0$ and those of angular motion from the identity $\iiint (x^i T_{j\nu}^\nu - x^j T_{i\nu}^\nu)\mathrm{d}v = 0$ neglecting all terms of order higher than the second. Since T_i^4, T_4^i, T_i^j contain no first order terms, the identity $\iiint T_{i\nu}^\nu \,\mathrm{d}v = 0$ reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint \dot{T}_i^4 \,\mathrm{d}v = -\frac{1}{2}\iiint W_i \nabla^2 W \,\mathrm{d}v\,.$$

Up to terms of order $\frac{3}{2}$ the expressions for T_i^4 and T_4^4 are (*)

$$T_i^4 = -W_{,i4} + \tfrac{1}{2}\gamma_{a,ai} - \tfrac{1}{2}\gamma_{i,aa} = -W_{,i4} + \tfrac{1}{2}(\mathrm{rot}\,\mathrm{rot}\,\gamma)_i = K^i\,, \tag{15}$$

$$T_4^4 = -\nabla^2 W = -8\pi\varrho \qquad (\gamma_i = g_{i4}). \tag{16}$$

([4]) V. A. Fock: *Journ. Exper. Theor. Phys. U.S.S.R.*, **1**, 81 (1939).

([5]) G. L. Clark: *Proc. Roy. Soc.*, **177**, 227 (1941).

(*) We denote the ordinary derivative of a quantity by a comma followed by the appropriate suffix.

The arbitrary functions K^i and ϱ appearing on the right hand side prevent the solution from vanishing identically and have non-zero values only inside the body. From Eqs. (15) and (16) we have

$$-8\pi\,\partial\varrho/\partial t = \operatorname{div} K\,, \tag{17}$$

whence

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint \varrho\,\mathrm{d}v = 0\,. \tag{18}$$

To proceed further we have to assume that the physical quantity represented by ϱ is strictly conserved and put $K^i = 8\pi\varrho\times$ velocity $= 8\pi\varrho U^i$. The plausibility of this assumption is evident from Eq. (18) and the form of the equation (17). Such an assumption is implied in all derivations of the equations of motion of a continuous body [5] and can be avoided only if matter is treated as a singularity in the field, that is, if ϱ is taken to be a Dirac δ-function. The classical equations of linear and angular motion of a body in a gravitational field are obtained as soon as this identification is made. For a dust of particles, on the other hand, they follow strictly from the field equations for empty space without any extra assumption. The surface integral conditions determining the two kinds of motion are

$$\iint l_k\left[-\delta_{ik}W_{,44}+\frac{1}{2}\gamma_{k,i4}-\frac{1}{2}\gamma_{i,k4}-\psi^2(\log\psi)_{ik}\right]\mathrm{d}S = 0\,,$$

$$\iint l_k\left[\delta_{jk}x^i\left(-W_{,44}+\frac{1}{2}\gamma_{a,a4}\right)-\frac{1}{2}x^i\gamma_{j,k4}+\frac{1}{2}\delta_{ik}\gamma_{j,4}-\right.$$

$$\left.-\psi^2(x^i\log\psi)_{jk}-\text{a similar expression with } i,\ j \text{ interchanged}\right]\mathrm{d}S = 0\,.$$

We thus see that the Newtonian equations of motion can be derived from very general considerations without solving Eqs. (15) and (16) for γ_i and W. These functions have a striking resemblence to the electromagnetic potentials and can be easily determined by imposing the co-ordinate condition

$$\operatorname{div}\gamma = 2W_{,4}$$

Eq. (15) then reduces to

$$\nabla^2\gamma = -16\pi\varrho U\,.$$

Solutions which do not satisfy this co-ordinate condition can be obtained by adding to γ the gradient of an arbitrary function.

RIASSUNTO (*)

Si discute la natura di un elemento lineare statico nella teoria della relatività e si dimostra che sia la forza che la torsione applicata a un corpo isolato debbono annullarsi affinchè nelle vicinanze del corpo si possa ottenere una soluzione statica delle equazioni di campo per il vuoto. Si mostra anche come con un procedimento per approssimazioni successive possa essere costruita una soluzione generale per un campo elettrogravitazionale statico.

(*) *Traduzione a cura della Redazione.*

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Application of the Chew-Low Formalism of Multi-Channel Reactions (*).

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Summary. — The Low equation for a non-relativistic scattering process is investigated, using a model in which the scattering source can exist in more than one state, and the « average » interaction between the source and the particle is factorizable. The branching ratio theorem and the optical theorem are derived directly from the Low equation. The Low equation is solved by the method of Castellejo, Dalitz and Dyson, and the unique solution is determined by investigating the spectrum of the unperturbed Hamiltonian. The resonance phenomena are studied and the Breit-Wigner resonance formula is derived utilizing the properties of the generalized R-function. The more general applicability of the method is indicated.

1. – Introduction.

The Chew-Low formalism (1) has been applied to a variety of problems (2) and it has been speculated that the R-function appearing in the Castellejo-Dalitz-Dyson solution (3) of the Low equation may in some way be related to the similar functions in the Wigner-Eisenbud theory of nuclear reactions (4).

(*) Supported in part by the U.S. Atomic Energy Commission.

(1) G. F. Chew and F. E. Low: *Phys. Rev.*, **101**, 1571 (1956).

(2) Suitably catalogued in J. G. Taylor: *Lectures on Dispersion Relations in Quantum Field Theory and Related Topics*, Lecture notes, University of Maryland (1958).

(3) L. Castellejo, R. H. Dalitz and F. J. Dyson: *Phys. Rev.*, **101**, 453 (1956), hereafter referred to as CDD.

(4) For an extensive bibliography, see A. M. Lane and R. G. Thomas: *Rev. Mod. Phys.*, **30**, 257 (1958). Also, reference (5).

To our knowledge, however, there has not been an application of the Chew-Low formalism to inelastic scattering, *i.e.*: to the case where the scattering source is capable of existing in more than one state, nor has it been explicitly demonstrated that the additive R-function leads to the results of the Wigner-Eisenbud theory of nuclear reactions.

Recently FESHBACH ([5]) has advanced a unified theory of nuclear reactions in which no assumption is made of sharp spatial cut-off of the interaction between the source and the scattered particle, and in which the dispersion theoretic character of the Breit-Wigner resonance formula is explicitly demonstrated. It appears possible therefore to derive Feshbach's results from the Low equation for inelastic scattering, at least for a class of theories for which the Low equation is soluble.

The Low equation is known to have an infinite manifold of solutions, as was shown by CDD, and the non-uniqueness of the solutions is attributed to the fact that the equation does not manifest the full physical content of the system ([6-8]). It has been argued that other properties would have to be utilized before a unique solution could be obtained. This point has recently been clarified by the works of HAAG ([9]), and of FAIRLIE and POLKINGHORNE ([10]), who show that the non-uniqueness is due to lack of information on the « hidden structure » of the system, and that the study of the energy spectrum of the unperturbed Hamiltonian suffices to choose the « physical » solution which is, of course, unique.

The purpose in the present work is twofold: firstly it is to construct a model of inelastic scattering for which the Low equation is exactly soluble, and to examine the consequences thereof; secondly it is to deduce the resonance behavior from the solution of the equation. The model we shall employ in our study assumes an arbitrary number of stable states of the source. The interactions between the source and the particle are assumed to be separable, *i.e.*: factorizable in the momentum representation. Our purpose in studying such a non-local interaction model is purely pedagogic. It is instructive to examine the general features of inelastic scattering through this simplified, soluble, albeit unrealistic model. We believe some important characteristics of inelastic scattering are not affected by the assumption of non-local interaction, in particular the resonance phenomena. This work is frankly motivated by the possibility of deriving the Breit-Wigner resonance formula from a dispersion theoretic consideration.

([5]) H. FESHBACH: *Ann. Phys.*, **5**, 357 (1958).
([6]) F. J. DYSON: *Phys. Rev.*, **100**, 157 (1957).
([7]) R. NORTON and A. KLEIN: *Phys. Rev.*, **109**, 584 (1958).
([8]) R. NORTON and A. KLEIN: *Phys. Rev.*, **109**, 991 (1958).
([9]) R. HAAG: *Nuovo Cimento*, **5**, 203 (1957).
([10]) D. B. FAIRLIE and J. C. POLKINGHORNE: *Nuovo Cimento*, **8**, 345, 555 (1958).

In Section 2 we describe our model and derive the Low equation. We then solve the Low equation by the method of CDD. It is shown by the method developed by HAAG that our model implies the solution *without* any additive R-function. It is shown in Section 3 that it is nonetheless possible to derive the Breit-Wigner resonance formula from the solution obtained in Section 2. Here we make use of the properties of the generalized R-function. The last Section contains discussions of our method and possible extensions thereof.

2. – The Low equation.

Consider, for example, a system of a nucleon and the target nucleus of mass number N. We may expand the wave function ψ of the system in the complete orthonormal set of the nucleus wave functions $u_\lambda(r_1, r_2, ..., r_N)$ where the subscript λ denotes collectively the quantum numbers which define a state of energy ε_λ, and the variables r_i, $i = 1, 2, ..., N$, include position, spin and isotopic spin variables:

$$\psi = \sum_\lambda \varphi_\lambda(r)\, u_\lambda(r_1, r_2, ..., r_N)\,, \tag{2.1}$$

where the coefficients $\varphi_\lambda(r)$ are functions of the variables of the incident nucleon. Inserting expansion (1) into the Schrödinger equation

$$[H_N = T + V(r; r_1, ..., r_N)]\psi = E\psi\,, \tag{2.2}$$

where H_N is the Hamiltonian for the target nucleus, so that

$$H_N u_\lambda = \varepsilon_\lambda u_\lambda\,, \tag{2.3}$$

T is the kinetic energy operator of the incident nucleon and V is the potential energy of the nucleon in the field of the nucleus, and utilizing the orthogonality of u_λ, we obtain a set of coupled equations for the amplitudes $\varphi_\lambda(r)$:

$$(T + \varepsilon_\alpha - E)\varphi_\alpha = -\sum_\lambda V_{\alpha\lambda}\varphi_\lambda\,, \tag{2.4}$$

where

$$\begin{cases} V_{\alpha\beta} = \int \mathrm{d}\boldsymbol{r}_1\, \mathrm{d}\boldsymbol{r}_2 \ldots \mathrm{d}\boldsymbol{r}_N\, u_\alpha^* V u_\beta = V_{\alpha\beta}(\boldsymbol{r})\,, \\ V_{\alpha\beta}(\boldsymbol{r}) = V_{\beta\alpha}^*(\boldsymbol{r})\,. \end{cases} \tag{2.5}$$

Equation (2.4) is then a convenient starting point for the discussion of nuclear reactions. However, this scheme suffers from the following two difficulties:

firstly expansion (2.1) is not suited for the discussion of exchange scattering or reactions such as (n, α) because the amplitudes φ_λ do not describe such channels in a manifest way; secondly, the identity of the incident nucleon with the nucleons in the nucleus is not taken into account in expansion (2.1). For the most part such difficulties can be overcome; for a full account of this point the reader is referred to Feshbach's paper.

Equation (2.4) suggests that we consider a field theoretic model whose Hamiltonian is given by (*)

$$H = H_0 + gH_I \tag{2.6}$$

and

$$H_0 = \sum_\lambda \int \mathrm{d}\boldsymbol{r}\, [\nabla\varphi_\lambda^*(\boldsymbol{r})\cdot\nabla\varphi_\lambda(\boldsymbol{r}) + \varepsilon_\lambda\varphi_\lambda^*\varphi_\lambda] , \tag{2.7}$$

$$\begin{cases} H_I = \sum_\lambda \int \mathrm{d}\boldsymbol{r} \int \mathrm{d}\boldsymbol{r}'\, \varphi_\lambda^*(\boldsymbol{r}) V_{\lambda\mu}(\boldsymbol{r}, \boldsymbol{r}')\varphi_\mu(\boldsymbol{r}') , \\ V_{\lambda\mu}(\boldsymbol{r}, \boldsymbol{r}') = V_{\mu\lambda}^*(\boldsymbol{r}', \boldsymbol{r}) , \end{cases} \tag{2.8}$$

where φ_λ is the field operator for the particle in the λ-th channel, ε_λ is the λ-th energy level of the source, and $V_{\mu\lambda}(\boldsymbol{r}, \boldsymbol{r}')$ is the average potential that the particle feels in transition from the λ-th channel to the μ-th channel. The equations of motion for the field operators φ_λ are, with local potentials $V_{\mu\lambda} = V_{\mu\lambda}(\boldsymbol{r})\,\delta(\boldsymbol{r} - \boldsymbol{r}')$, given by Eq. (2.4). The number of energy levels of the source may be finite or infinite. For the sake of brevity of notation, we assume that the energy levels are discrete.

In the momentum space, the Hamiltonian is written as

$$\begin{cases} H_0 = \sum_\lambda \int \mathrm{d}\boldsymbol{p} [\boldsymbol{p}^2 + \varepsilon_\lambda] a_\lambda^*(p) a_\lambda(p) , \\ H_I = \sum_{\lambda\mu} \int \mathrm{d}\boldsymbol{p} \int \mathrm{d}\boldsymbol{p}'\, a_\lambda^*(\boldsymbol{p}) V_{\lambda\mu}(\boldsymbol{p}, \boldsymbol{p}')\, a_\mu(\boldsymbol{p}') , \end{cases} \tag{2.9}$$

where

$$\begin{cases} a_\lambda(p) = \dfrac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}\boldsymbol{r}\, \varphi_\lambda(\boldsymbol{r}) \exp[-i\boldsymbol{p}\cdot\boldsymbol{r}] , \\ V_{\lambda\mu}(\boldsymbol{p}, \boldsymbol{q}) = \dfrac{1}{(2\pi)^3} \int \mathrm{d}^3\boldsymbol{r} \int \mathrm{d}^3\boldsymbol{q}'\, V_{\lambda\mu}(\boldsymbol{r}, \boldsymbol{r}') \exp[i\boldsymbol{p}\cdot\boldsymbol{r}] \exp[i\boldsymbol{q}\cdot\boldsymbol{r}'] . \end{cases} \tag{2.10}$$

To simplify our analysis, we further assume that $V_{\lambda\mu}(\boldsymbol{p}, \boldsymbol{p}')$ are separable:

$$V_{\lambda\mu}(\boldsymbol{p}, \boldsymbol{p}') = v_\lambda(p) v_\mu^*(p') \tag{2.11}$$

(*) We use the units $\hbar = 2m = 1$.

and that $v_\lambda(p)$ is spherically symmetric. The last condition implies that only the S-waves participate in scattering (elastic and inelastic).

By applying the methods of WICK [11], we readily obtain for the S-matrix:

$$(2.12)\qquad (p_\beta, \beta\,|\,S\,|\,p_\alpha, \alpha) = \delta_{\alpha\beta}\,\delta(\boldsymbol{p}_\beta - \boldsymbol{p}_\alpha) - 2\pi i\,\delta(\varepsilon_\alpha - \varepsilon_\beta + p_\alpha^2 - p_\beta^2)(p_\beta, \beta\,|\,T\,|\,p_\alpha, \alpha)\,,$$

where α, β denote the initial and final channels, respectively, $\boldsymbol{p}_\alpha$ and $\boldsymbol{p}_\beta$ are the corresponding asymptotic momenta, and the scattering amplitude is given by

$$(2.13)\qquad (p_\beta, \beta\,|\,T\,|\,p_\alpha, \alpha) = g(p_\beta, \beta\,|\,H_I\,|\,p_\alpha, \alpha) + g^2\left(p_\beta, \beta\left|H_I\,\frac{1}{E + i\eta - H}\,H_I\right|p_\alpha, \alpha\right) =$$

$$= g v_\beta(p_\beta) v_\alpha^*(p_\alpha)\left[1 + g\left(0\left|V\,\frac{1}{E + i\eta - H}\,V^*\right|0\right)\right],$$

where

$$V = \sum_\lambda \int \mathrm{d}\boldsymbol{p}\, v_\lambda^*(p) a_\lambda(p)\,,$$

and $|0)$ denotes the vacuum state. On the other hand, we also have

$$(2.14)\qquad (p_\beta, \beta\,|\,T\,|\,p_\alpha, \alpha) = g(\Psi^{(-)}_{\beta,p_\beta}, H_I a_\alpha^*(p_\alpha)\Psi_0) = g(\Psi^{(-)}_{\beta,p_\beta}, V^*\Psi_0)\, v_\alpha^*(p_\alpha)\,,$$

where Ψ_0 is the vacuum state and $\Psi^{(-)}_{\beta,p_\beta}$ is the incoming wave state vector given by

$$(2.15)\qquad \Psi^{(-)}_{\beta,p_\beta} = a_\beta^*(p_\beta)\Psi_0 + g\,\frac{1}{E - i\eta - H_0}\,H_I \Psi^{(-)}_{\beta,p_\beta}\,.$$

If we denote the elastic scattering amplitude in the λ-th channel by $\varrho_\lambda(p)$,

$$(2.16)\qquad \varrho_\lambda(p) \equiv (p, \lambda\,|\,T\,|\,p, \lambda) = g(\Psi^{(-)}_{\lambda,p}, V^*\Psi_0) v_\lambda^*(p)\,,$$

then, in general, the scattering amplitude $(p, \lambda\,|\,T\,|\,q, \mu)$ can be written as

$$(2.17)\qquad \begin{cases} (p, \lambda\,|\,T\,|\,q, \mu) = g(\Psi^{(-)}_{\lambda,p}, V^*\Psi_0) v_\mu^*(q)\,, \\ \qquad\qquad = \varrho_\lambda(p)\,\dfrac{v_\mu^*(q)}{v_\lambda^*(p)}\,. \end{cases}$$

By virtue of Eq. (2.17), we rewrite Eq. (2.13) as

$$(2.18)\qquad \frac{\varrho_\lambda(p)}{g\,|\,v_\lambda(p)\,|^2} = 1 + g\sum_\mu \int \frac{\mathrm{d}^3 q}{\varepsilon_\lambda + p^2 + i\eta - \varepsilon_\mu - q^2}\left|\frac{\varrho_\mu(q')}{g v_\mu(q)}\right|^2.$$

[11] G. C. WICK: *Rev. Mod. Phys.*, **27**, 339 (1955).

If we define the function $h_\lambda^{(+)}(t)$, where $t = p^2$,

$$h_\lambda^{(+)}(t) \equiv \frac{\varrho_\lambda(p)}{g\,|v_\lambda(p)|^2}\,, \tag{2.19}$$

and use it to rewrite Eq. (2.18), we obtain

$$h_\lambda^{(+)}(t) = 1 + g\sum_\mu \int_0^\infty \frac{2\pi p'\,\mathrm{d}t'}{\varepsilon_\lambda + t + i\eta - \varepsilon_\mu - t'}\,|v_\mu(p')|^2\,|h_\mu^{(+)}(t')|^2\,, \tag{2.20}$$

which is the Low equation.

For later discussion it is convenient to define the function $\mathcal{C}_{\beta\alpha}(E)$:

$$\mathcal{C}_{\beta\alpha}(E) = -2\pi^2(p_\beta p_\alpha)^{\frac{1}{2}}(p_\beta, \beta\,|\,T\,|\,p_\alpha, \alpha) = \tag{2.21}$$

$$= -2\pi^2(p\ p_\alpha)^{\frac{1}{2}} g v_\beta(p_\beta) v_\alpha^*(p_\alpha) h_\beta^{(+)}(p_\beta^2)\,,$$

where $E = \varepsilon_\beta + p_\beta^2 = \varepsilon_\alpha + p_\alpha^2$.

The total cross-section $\sigma_{\beta\alpha}(E)$ is given by

$$\sigma_{\beta\alpha}(E) = \frac{4\pi}{p_\alpha^2}\,|\mathcal{C}_{\beta\alpha}|^2\,. \tag{2.22}$$

A property of the function $h_\lambda^{(+)}(t)$ that can be read off immediately from Eq. (2.20) is

$$h_\lambda^{(+)}(t) = h_\mu^{(+)}(t + \varepsilon_\lambda - \varepsilon_\mu)\,. \tag{2.23}$$

The physical significance of Eq. (2.23) becomes clear if one re-expresses it in the form

$$\left|\frac{\mathcal{C}_{\lambda\alpha}(E)}{\mathcal{C}_{\mu\alpha}(E)}\right|^2 = \frac{p_\lambda}{p_\mu}\left|\frac{v_\lambda(p_\lambda)}{v_\mu(p_\mu)}\right|^2\,, \tag{2.24}$$

where $E = \varepsilon_\lambda + p_\lambda^2 = \varepsilon_\mu + p_\mu^2$.

Eq. (2.24) gives the branching ratio of two reactions, $\alpha \to \lambda$, and $\alpha \to \mu$. A remarkable feature of Eq. (2.24) is that the branching ratio is independent of the incident channel α, because α does not appear on the right hand side of Eq. (2.24). It is characteristic of our separable interaction model that the incoming particle « loses its memory » after it has been absorbed by the source.

We now proceed to obtain the solution of the Low equation (2.20). To this end, we extend the argument of Eq. (2.20), $t + i\eta$, into the complex Z-plane,

and thereby obtain the function $h_\lambda(Z)$:

$$h_\alpha(Z) = 1 + g \sum_\lambda \int_0^\infty \frac{2\pi p' \mathrm{d}p'}{\varepsilon_\alpha + Z - \varepsilon_\lambda - t'} |v_\lambda(p')|^2 |h_\lambda^{(+)}(t')|^2 , \tag{2.25}$$

$h_\alpha(Z)$ obviously satisfies

$$\lim_{\eta \to 0_+} h_\alpha(t + i\eta) = h_\alpha^{(+)}(t) .$$

The relevant properties of $h_\alpha(Z)$ are as follows:

a) $h_\alpha(Z)$ is analytic everywhere in the complex Z-plane except on the real axis from $\varepsilon_0 - \varepsilon_\alpha$ (where ε_0 is the ground state energy level of the source) to ∞.

b) On the real axis, $h_\alpha(Z)$ has branch points at $t = \varepsilon_\lambda - \varepsilon_\alpha$ for all λ. The branch line of the term

$$\int_0^\infty \frac{2\pi p' \, \mathrm{d}t'}{Z + \varepsilon_\alpha - \varepsilon_\lambda - t'} |v_\lambda(p')|^2 |h_\lambda^{(+)}(t')|^2 ,$$

on the right side of Eq. (2.25) extends from

$$t = \varepsilon_\lambda - \varepsilon_\alpha \qquad \text{to} \qquad + \infty .$$

c) $h_\alpha(Z) \to 1$ as $Z \to \infty$.

d) The imaginary part of $h_\alpha(Z)$ is given by

$$\operatorname{Im} h_\alpha(Z) = -g \left(\sum_\lambda \int_0^\infty \frac{2\pi p' \, \mathrm{d}t'}{|Z + \varepsilon_\alpha - \varepsilon_\lambda - t'|^2} |v_\lambda(p')|^2 |h_\lambda^{(+)}(t')|^2 \right) \operatorname{Im} Z ,$$

i.e.: $h_\alpha(Z)$ is a generalized R-function in the sense of CDD [3].

e) Along the branch lines extending from $t = \varepsilon_0 - \varepsilon_\alpha$ to $+\infty$, $h_\alpha(Z)$ has a discontinuity given by

$$\lim_{\eta \to 0_+} [h_\alpha(t + i\eta) - h_\alpha(t - i\eta)] = 2i \operatorname{Im} h_\alpha^{(+)}(t) = \tag{2.26}$$

$$= -4\pi^2 g i \sum_\lambda \int_0^\infty p' \, \mathrm{d}t' \, \delta(\varepsilon_\alpha + t - \varepsilon_\lambda - t') |v_\lambda(p')|^2 |h_\lambda^{(+)}(t')|^2 =$$

$$= -4\pi^2 g i \sum_{\lambda:\ \text{open}} p_\lambda |v_\lambda(p_\lambda)|^2 |h_\lambda^{(+)}(p_\lambda^2)|^2 .$$

where

$$p_\lambda = \sqrt{\varepsilon_\alpha + t - \varepsilon_\alpha} .$$

Because of the energy delta functions $\delta(\varepsilon_\alpha + t - \varepsilon_\lambda - t')$ under the integrals, summation is to be taken only over the energetically open channels, *i.e.* channels with ε_λ, $\varepsilon_\lambda \leqslant \varepsilon_\alpha + t$.

Equation (2.26) amounts to a generalized optical theorem. Rewriting Eq. (2.26) as

$$\text{Im}\, h_\alpha^{(+)}(p) = -2\pi^2 g \sum_{\lambda:\text{open}} p_\lambda |v_\lambda(p_\lambda)|^2 |h_\lambda^{(+)}(p_\lambda)|^2 \tag{2.27}$$

and multiplying Eq. (2.27) by $-2\pi^2 g p_\alpha |v_\alpha(p)|^2$, we obtain

$$\begin{cases} \text{Im}\, \mathcal{C}_{\alpha\alpha}(E) = \sum\limits_{\lambda:\text{ open}} |\mathcal{C}_{\lambda\alpha}(E)|^2 \\ \text{or} \\ \dfrac{4\pi}{p_\alpha^2} \text{Im}\, \mathcal{C}_{\alpha\alpha}(E) = \sum\limits_{\lambda:\text{ open}} \sigma_{\lambda\alpha}(E)\,. \end{cases} \tag{2.28}$$

The above observations *a*)-*e*) enable us to write down $h_\alpha^{-1}(Z)$ by the method of CDD:

$$\frac{1}{h_\alpha(Z)} = 1 + g \sum_\lambda \int_{\varepsilon_\lambda - \varepsilon_\alpha}^{\infty} \frac{2\pi\sqrt{\varepsilon_\alpha + t' - \varepsilon_\lambda}\, dt'}{t' - Z} |v_\lambda(\sqrt{\varepsilon_\alpha - \varepsilon_\lambda - t'})|^2 - g \sum_i \frac{g\Lambda_i^{(\alpha)}}{Z - t_i^{(\alpha)}} = \tag{2.29}$$

$$= 1 + g \sum_\lambda \int_0^\infty \frac{4\pi p'^2\, dp'}{\varepsilon_\lambda + p'^2 - \varepsilon_\alpha - Z} |v_\lambda(p')|^2 - \sum_i \frac{g\Lambda_i^{(\alpha)}}{Z - t_i^{(\alpha)}}\,.$$

In deriving Eq. (2.29), use in made of Eq. (2.23). It can be shown that the property *e*) of $h_\alpha(Z)$ implies $\Lambda_i^{(\alpha)} \geqslant 0$.

We seemingly have obtained a manifold of solutions because of the indeterminacy of the meromorphic R-function

$$R^{(\alpha)} = -\sum_i \frac{g\Lambda_i^{(\alpha)}}{Z - t_i^{(\alpha)}}\,,$$

on the right hand side of Eq. (2.29). The solution of Eq. (2.20) *corresponding to the Hamiltonian* (2.6)–(2.8) is, however, uniquely given by

$$\frac{1}{h_\alpha^{(+)}(t)} = 1 + g \sum_\lambda J_\lambda^{(+)}(\varepsilon_\alpha + t)\,, \tag{2.30}$$

where

$$J_\lambda^{(+)}(E) = \int_0^\infty \frac{4\pi p^2\, dp\, |v_\lambda(p)|^2}{\varepsilon_\lambda + p^2 - E - i\eta}\,, \tag{2.31}$$

without the additive meromorphic R-function $R^{(\alpha)}$. The proof of this assertion follows from the fact that in our model the asymptotic forms of the wave

functions $\Psi^{(-)}_{\alpha,p}$ form a complete set of eigenfunctions of the unperturbed Hamiltonian H_0:

$$\lim_{t\to+\infty} \exp[iH_0t]\exp[-iHt]\Psi^{(-)}_{\alpha,p} = a^*_\alpha(p)\Psi_0 \equiv |p,\alpha\rangle\,, \tag{2.32}$$

$$\sum_\alpha \int \mathrm{d}^3p\,|p,\alpha\rangle\langle p,\alpha| = 1\,, \tag{2.33}$$

and there are no « hidden structures » *i.e.*, eigenstates of H_0 which cannot be obtained by the limiting process of Eq. (2.32).

It follows from the hermiticity of H_I that

$$B(\beta,q;\alpha,p) = B^*(\alpha,p;\beta,q)\,, \tag{2.34}$$

where

$$B(\beta,q;\alpha,p) = (\Psi^{(-)}_{\beta,q}, H_I\Psi^{(-)}_{\alpha,p})\,. \tag{2.35}$$

Substituting Eq. (2.15) into Eq. (2,35), we obtain

$$B(\beta,q;\alpha,p) = g(\Psi^{(-)}_{\beta,q}, V^*\Psi_0)v^*_\lambda(p) + g^2\left(\Psi^{(-)}_{\beta,q}, H_I\frac{1}{\varepsilon_\alpha + p^2 - i\eta - H_0}H_I\Psi^{(-)}_{\alpha,p}\right). \tag{2.36}$$

Now using the « asymptotic completeness » condition, (2,33), and notations of Eqs. (2.16), (2.17) and (2.19), we find (*)

$$B(\beta,q;\alpha,p) = gv_\beta(q)h^{(+)}_\beta(q^2)v^*_\lambda(p)h^{(+)*}_\lambda(p^2)\left[\frac{1}{h^{(+)*}_\lambda(p^2)} - g\sum_\lambda \int_0^\infty \frac{4\pi p'^2\mathrm{d}p'\,|v_\lambda(p')|^2}{\varepsilon_\lambda + p'^2 - \varepsilon_\lambda - p^2 + i\eta}\right]. \tag{2.37}$$

Substitution of Eq. (2.29) into Eq. (2.37) yields

$$B(\beta,q;\alpha,p) = gv_\beta(q)h^{(+)}_\beta(q^2)v^*_\alpha(p)h^{(+)*}_\alpha(p^2)\left[1 - \sum_i \frac{g\Lambda^{(\alpha)}_i}{p^2 - l^{(\alpha)}_i}\right]. \tag{2.38}$$

This expression can satisfy the hermiticity condition, Eq. (2.34), only if all $\Lambda^{(\alpha)}_i$ vanish. This completes the proof.

If there were hidden structures in our model, as in the Dyson model [6] or in the Norton-Klein model [8] so that we had

$$\sum_\alpha \left[\int \mathrm{d}^3p\,|p,\alpha\rangle\langle p,\alpha| + \sum_i |\,|\alpha,i\rangle\langle\alpha,i|\right] = 1\,,$$

(*) We are here deriving essentially what Fairlie and Polkinghorne term the reciprocal Low equation. We could have derived Eq. (2.30) directly by the same procedure and thereby avoided the ambiguity concerning the additive meromorphic R-function. We feel, nonetheless, it more instructive to emphasize the criterion for the « physical » solution, especially in view of the controversy that existed in the past.

instead of Eq. (2.33), where $|\alpha, i\rangle$ are the eigenstates of H_0 in the α-th channel which cannot be obtained by the limiting process of Eq. (2.32), then Eq. (2.37) would have read

$$B(\beta, q; \alpha, p) = g v_\beta(q) v_\alpha^*(p) h_\beta^{(+)}(q^2) h_\alpha^{(+)*}(p^2) \cdot$$

$$\cdot \left\{ \frac{1}{h_\alpha^{(+)*}(p^2)} - g \sum_\lambda \left[\int_0^\infty \frac{4\pi p'^2 \mathrm{d}p' |v_\lambda(p')|^2}{\varepsilon_\lambda + p'^2 - \varepsilon_\alpha - p^2 + i\eta} + \sum_i \frac{4\pi p_i^2 |v_\lambda(p_i)|^2}{\varepsilon_\lambda + p_i^2 - \varepsilon_\alpha - p^2} \right] \right\}^{-1},$$

where $E_i^{(\lambda)} = \varepsilon_\lambda + p_i^2$ is the energy level of the hidden structure $|\lambda, i,\rangle$ and the « physical » solution would have been

$$\frac{1}{h_\alpha^{(+)}(p^2)} = 1 + g \sum_\lambda \left[\int_0^\infty \frac{4\pi p'^2 \mathrm{d}p' |v_\lambda(p')|^2}{\varepsilon_\lambda + p'^2 - \varepsilon_\alpha - p^2 - i\eta} - \sum_i \frac{4\pi p_i^2 |v_\lambda(p_i)|^2}{p^2 + \varepsilon_\alpha - E_i^{(\lambda)}} \right].$$

3. – Resonance scattering.

In the previous Section, we have established that

$$\mathcal{T}_{\beta\alpha}(E) = -2\pi^2 (p_\beta p_\alpha)^{\frac{1}{2}} g v_\beta(p_\beta) v_\alpha^*(p_\alpha) [1 + g \sum_\lambda J_\lambda^{(+)}(E)]^{-1}. \tag{3.1}$$

In this form, the resonance behavior of the scattering amplitudes characteristic of many-channel processes is not demonstrated explicitly. We shall devote ourselves in this section to the study of the narrow resonances which are described by the Breit-Wigner formula. The point of view to be taken on the mechanism of resonance is essentially that of FESHBACH [5]. We view the resonances as occuring when the total energy of the system is close to the energy level of a bound state of the system which remains when the interactions connecting different channels are « switched off ». Such bound states can be shown to become radioactive, when the interactions coupling different channels are « turned on » [12].

We first note that $1 + g J_\lambda^{(+)}(E)$ which appears in Eq. (3.1) is directly related to the reciprocal of the elastic scattering amplitude in the λ-th channel when the channels are uncoupled. To be more precise, let us consider the « diagonal » Hamiltonian $\mathcal{H}$:

$$\begin{cases} \mathcal{H} = \mathcal{H}_0 + g\mathcal{H}_I, \\ \mathcal{H}_0 = \sum_\lambda \int \mathrm{d}^3 p (p^2 + \varepsilon_\lambda) a_\lambda^*(p) a_\lambda(p), \\ \mathcal{H}_I = \sum_\lambda \int \mathrm{d}^3 p \, v_\lambda(p) a_\lambda^*(p) \int \mathrm{d}^3 p' v_\lambda^*(p') a_\lambda(p'). \end{cases} \tag{3.2}$$

[12] B. ZUMINO: *On the Formal Theory of Collision and Reaction Processes*, New York University Institute of Mathematical Sciences Research Report PB122994 (1956).

The diagonal Hamiltonian $\mathcal{H}$ describes the state of affairs when the interactions coupling different channels, $V_{\mu\lambda}$, $\mu \neq \lambda$, of Eq. (2.8) are « switched off ». The incoming wave state vector $\chi^{(-)}_{\lambda,p}$ is given by

$$\chi^{(-)}_{\lambda,p} = a^*_\lambda(p)\Psi_0 + g\,\frac{1}{E_\lambda + p^2 - i\eta - \mathcal{H}_0}\,\mathcal{H}_I\,\chi^{(-)}_{\lambda,p}\,. \tag{3.3}$$

There is no inelastic scattering when the channels are uncoupled, and the elastic scattering amplitude in the λ-th channel is given by

$$g(\chi^{(-)}_{\lambda,p},\,\mathcal{H}_I a^*_\lambda(p)\Psi_0) = \\ = g(\chi^{(-)}_{\lambda,p},\,\mathcal{H}_I\chi^{(-)}_{\lambda,p}) + g^2\left(\chi^{(-)}_{\lambda,p},\,\mathcal{H}_I\,\frac{1}{\mathcal{H}_0 - \varepsilon_\lambda - p^2 + i\eta}\,\mathcal{H}_I\chi^{(-)}_{\lambda,p}\right). \tag{3.4}$$

With the notation

$$\tau_\lambda(p) = g(\chi^{(-)}_{\lambda,p},\,\mathcal{H}_I a^*_\lambda(p)\Psi_0) = g(\chi^{(-)}_{\lambda,p},\,V^*_\lambda\Psi_0)v^*_\lambda(p)\,, \tag{3.5}$$

where

$$V^*_\lambda = \int \mathrm{d}^3q\, v_\lambda(q)\, a^*_\lambda(q)\,,$$

Eq. (3.4) can be shown to lead to

$$\tau_\lambda(p) = \frac{1}{g}\left|\frac{T_\lambda(p)}{v_\lambda(p)}\right|^2\left[1 + g\int_0^\infty \frac{4\pi p'^2\,\mathrm{d}p'\,|\,v_\lambda(p')\,|^2}{p'^2 - p^2 + i\eta}\right],$$

or

$$\frac{\tau_\lambda(p)}{g\,|\,v_\lambda(p)\,|^2} = [1 + gJ^{(+)}_\lambda(\varepsilon_\lambda + p^2)]^{-1}. \tag{3.6}$$

We now assume that the coupling constant g is negative. Then for certain $\mathcal{E}_\lambda$, $-\infty < \mathcal{E}_\lambda < \varepsilon_\lambda$,

$$1 + gJ^{(+)}_\lambda(\mathcal{E}_\lambda) = 1 + g\int_0^\infty \frac{\mathrm{d}^3p'\,|\,v_\lambda(p)'\,|^2}{p^2 + \varepsilon_\lambda - \mathcal{E}_\lambda} = 0\,, \tag{3.7}$$

provided that

$$g\int \frac{\mathrm{d}^3p}{p^2}\,|\,v_\lambda(p)\,|^2 < -1\,, \tag{3.8}$$

since the function $J^{(+)}_\lambda(E)$ is monotonically increasing in the range $-\infty < E < \varepsilon_\alpha$. It also follows from the above remark that there is only one zero $\mathcal{E}_\lambda$ of Eq. (3.7).

Eq. (3.7) in fact is the eigenvalue equation for the Schrödinger equation implied by the diagonal Hamiltonian (3.2), and the root of Eq. (3.7), $\mathcal{E}_\lambda$, is the energy level of the bound state in the uncoupled λ-channel, as one may readily verify either by solving the Schrödinger equation, or by noting that Eq. (3.7) is the Fredholm determinant for the Lippmann-Schwinger integral equation (3.3). We further assume that $1 + gJ_\lambda^{(+)}(E)$ does not have zeroes in the range $\varepsilon_\lambda \leqslant E < \infty$:

$$1 + gJ_\lambda^{(+)}(E) = 1 + g \cdot \int \frac{\mathrm{d}^3 p \, |v_\lambda(p')|^2}{p' + \varepsilon_\alpha - E - i\eta} \neq 0 \qquad \text{for } E \geqslant \varepsilon_\lambda, \text{ for all } \lambda. \tag{3.9}$$

This condition on the potential $v_\lambda(p)$ is necessary in order that the wave function is given uniquely in the continuum $\varepsilon_\lambda \leqslant E < \infty$.

Returning to the discussion of inelastic scattering, we suppose that the total energy of the system $E = \varepsilon_\alpha + p^2$ is very close to the energy level $\mathcal{E}_\gamma$ of the bound state of the uncoupled γ-channel. To make the singularity of $[1+gJ_\gamma^{(+)}(E)]^{-1}$ at $E = \mathcal{E}_\gamma$ explicit we re-express $[1+gJ_\gamma^{(+)}(E)]^{-1}$ by the method of CDD. We extend the argument $E + i\eta$ of $J_\gamma^{(+)}(E)$ into the complex Z-plane and thereby define the function $J_\gamma(Z)$:

$$1 + gJ_\gamma(Z) \equiv 1 + g\int_0 \frac{\mathrm{d}^3 p \, |v_\gamma(p')|^2}{\varepsilon_\gamma + p'^2 - Z} \,. \tag{3.10}$$

It is sufficient to note that $1+gJ_\gamma(Z)$ is a generalized R-function whose imaginary part has the same sign as that of g in the upper half-plane, approaches 1 as Z goes to infinity and has a branch line on the real axis for $\varepsilon_\gamma \leqslant Z < +\infty$, with a discontinuity given by

$$\lim_{\eta \to 0_+} [gJ_\gamma(E + i\eta) - g\,J_\gamma(E - i\eta)] = 2ig \operatorname{Im} J_\gamma^{(+)}(E) = \\ = 4\pi^2 g i \sqrt{E - \varepsilon_\gamma}\, |v_\gamma(\sqrt{E - \varepsilon_\gamma})|^2 \,, \tag{3.11}$$

and has a zero at $E = \mathcal{E}_\gamma$, $\mathcal{E}_\gamma < \varepsilon_\gamma$. Hence we have

$$\frac{1}{1 + gJ_\gamma^{(+)}(E)} = 1 + g\left[\frac{C_\gamma}{E - \mathcal{E}_\gamma} + \int \frac{\mathrm{d}^3 p'}{E + i\eta - p'^2 - \varepsilon_\gamma} \left| \frac{v_\gamma(p')}{1 + gJ_\gamma^{(+)}(p'^2 + \varepsilon_\gamma)} \right|^2 \right], \tag{3.12}$$

where $C_\gamma > 0$.

Substituting Eq. (3.12) into Eq. (3.1), we obtain

$$\mathcal{T}_{\beta\alpha}(E) = -2\pi^2 (p_\beta\, p_\alpha)^{\frac{1}{2}} v_\beta(p)_\beta v_\alpha^*(p_\alpha) g \cdot \\ \cdot \left[\frac{1}{1 + g(C_\gamma/(E - \mathcal{E}_\gamma)) + gD_\gamma(E)} + g \sum_{\lambda \neq \gamma} J_\lambda^{(+)}(E)\right]^{-1}, \tag{3.13}$$

where $E = \varepsilon_\alpha + p_\alpha^2 = \varepsilon_\beta + p_\beta^2$, and

$$D_\gamma(E) = \int \frac{\mathrm{d}^3 p'}{E + i\eta - p'^2 - \varepsilon_\gamma} \left| \frac{v_\gamma(p')}{1 + g J_\gamma^{(+)}(\varepsilon_\gamma + p'^2)} \right|^2 .$$

After some algebraic manipulations, one finds

$$\mathcal{T}_{\beta\alpha}(E) = -2\pi^2 (p_\beta p_\alpha)^{\frac{1}{2}} v_\beta(p_\beta) v_\alpha^*(p_\alpha) g \cdot \left\{ \frac{1 + g D_\gamma}{(1 + g D_\gamma) g \sum_{\lambda \neq \gamma} J_\lambda^{(+)} + 1} + \right. \tag{3.14}$$

$$\left. + \frac{1}{[(1 + g D_\gamma) g \sum_{\lambda \neq \gamma} J_\lambda^{(+)} + 1]^2} \frac{g C_\gamma}{(E - \mathcal{E}_\gamma) + \dfrac{g^2 C_\gamma \sum_{\lambda \neq \gamma} J_\lambda^{(+)}}{(1 + g D_\gamma) g \sum_{\lambda \neq \gamma} J_\lambda^{(+)} + 1}} \right\} .$$

We shall consider the following two cases separately:

1) *Resonance in pure elastic scattering.* We shall treat the case where all channels are energetically closed except the incident channel, α, so that inelastic scattering is not possible. Then $E = p_\alpha^2 + \varepsilon_\alpha < \varepsilon_\lambda$ for all $\lambda > \alpha$. Hence

$$\operatorname{Im} J_\lambda^{(+)}(E) = \operatorname{Im} \int_0^\infty \frac{2\pi p' \, \mathrm{d}t' \, |v_\lambda(p')|^2}{\varepsilon_\lambda + t' - E - i\eta} = \begin{cases} 2\pi^2 p \, |v_\alpha(p)|^2 & \lambda = \alpha \\ 0 & \lambda \neq \alpha \end{cases} \tag{3.15}$$

and

$$\operatorname{Im} D_\gamma(E) = 0 , \qquad (\gamma \neq \alpha) \tag{3.16}$$

If we define the potential scattering phase shift δ_p by

$$\exp[-i\delta_p] = \frac{(1 + g D_\gamma) g \sum_{\lambda \neq \gamma} J_\lambda^{(+)} + 1}{|(1 + g D_\gamma) g \sum_{\lambda \neq \gamma} J_\lambda^{(+)} + 1|} , \tag{3.17}$$

then

$$\sin \delta_p = -\operatorname{Im} \exp[-i\delta_p] = \frac{-2\pi^2 p (1 + g D_\gamma) g \, |v_\alpha|^2}{|(1 + g D_\gamma) g \sum_{\lambda \neq \gamma} J_\lambda^{(+)} + 1|} , \tag{3.18}$$

and Eq. (3.14) can be written as

$$\mathcal{T}_{\alpha\alpha}(E) = \exp[i\delta_p] \sin \delta_p - \exp[2i\delta_p] \frac{\dfrac{2\pi^2 p g^2 C_\gamma \, |v_\alpha(p)|^2}{[(1 + g D_\gamma) \sum_{\lambda \neq \gamma} J_\lambda^{(+)} + 1]^2}}{(E - \mathcal{E}_\gamma) + \dfrac{g^2 C_\gamma \sum_{\lambda \neq \gamma} J_\lambda^{(+)}}{(1 + g D_\gamma) \sum_{\lambda \neq \gamma} J_\lambda^{(+)} + 1}} . \tag{3.19}$$

The imaginary part of the resonance denominator is

$$\mathrm{Im}\,\frac{g^2C_\gamma\sum_{\lambda\neq\gamma}J_\lambda^{(+)}}{(1+gD_\gamma)g\sum_{\lambda\neq\gamma}J_\lambda^{(+)}+1}=\frac{2\pi^2pgC_\gamma|v_\alpha(p)|^2}{|(1+gD_\gamma)\sum_{\lambda\neq\gamma}gJ_\lambda^{(+)}+1|^2}\,.$$

This suggests the definition of the level width Γ^γ as

$$\Gamma^\gamma\equiv 2\,\mathrm{Im}\,\frac{g^2C_\gamma\sum_{\lambda\neq\gamma}J_\lambda^{(+)}}{(1+gD_\gamma)g\sum_{\lambda\neq\gamma}J_\lambda^{(+)}+1}=pC_\gamma\left|\frac{2\pi g v_\alpha(p)}{(1+gD_\gamma)g\sum_{\lambda\neq\gamma}J_\lambda^{(+)}+1}\right|^2, \tag{3.20}$$

and the energy level shift Δ_γ as

$$\Delta_\gamma\equiv\mathrm{Re}\,\frac{g^2C_\gamma\sum_{\lambda\neq\gamma}J_\lambda^{(+)}}{(1+gD_\gamma)\sum_{\lambda\neq\gamma}J_\lambda^{(+)}+1}\,. \tag{3.21}$$

Then in virtue of Eqs. (3.17), (3.18), (3.20), and (3.21), Eq. (3.19) can be written as

$$\mathcal{C}_{\alpha\alpha}(E)=\left[\exp[i\delta_p]\sin\delta_p-\exp[2i\delta_p]\,\frac{\frac{1}{2}\Gamma^\gamma}{(E-\mathcal{E}_\gamma+\Delta_\gamma)+\frac{1}{2}\Gamma^\gamma}\right]. \tag{3.22}$$

This is the Breit-Wigner one-level formula. The unaesthetic appearance of Γ^γ and Δ_γ in Eqs. (3.20) and (3.21) are somewhat frightening. In Feshbach's theory, however, such complicated natures of Γ^γ and Δ_γ are made implicit by the use of wave functions ($v_0^{(+)}$, $v_0^{(-)}$ in his notation) which describe the motion of a particle in a complicated, energy dependent potential (U_n of his Eq. (2.28)).

2) *Resonance in inelastic scattering.* Suppose that more than one channel is energetically open and the total energy is close to $\mathcal{E}_\gamma$, where the γ-channel is closed, *i.e.* $\varepsilon_\gamma>E$. We define $\mathcal{C}_{\beta\alpha}^{(p)}(E)$ by

$$\mathcal{C}_{\beta\alpha}^{(p)}(E)=-2\pi^2(p_\beta p_\alpha)^{\frac{1}{2}}v_\beta(p_\beta)v_\alpha^*(p_\alpha)\,\frac{1+gD_\gamma}{(1+gD_\gamma)\sum_{\lambda\neq\gamma}J_\lambda^{(+)}+1}\,. \tag{3.23}$$

With this definition Eq. (3.19), can now be written as

$$\mathcal{C}_{\beta\alpha}(E)=\mathcal{C}_{\beta\alpha}^{(p)}(E)-\exp[+2i\xi]\,\frac{\dfrac{2\pi^2(p_\beta p_\alpha)^{\frac{1}{2}}gC_\gamma v_\beta(p_\beta)v_\alpha^*(p_\alpha)}{[|(1+gD_\gamma)g\sum_{\lambda\neq\gamma}J_\lambda^{(+)}+1|]^2}}{(E-\mathcal{E}_\gamma)+\dfrac{g^2C_\gamma\sum_{\lambda\neq\gamma}J_\lambda^{(+)}}{[|(1+gD_\gamma)g\sum_{\lambda\neq\gamma}J_\lambda^{(+)}+1|]^2}}\,, \tag{3.24}$$

where

$$\xi = -\operatorname{Arg}\{(1+gD_\gamma)g\sum_{\lambda\neq\gamma} J_\lambda^{(+)}+1\}.$$

The imaginary part of the resonance denominator is given by

$$\operatorname{Im}\frac{g^2C_\gamma\sum_{\lambda\neq\gamma}J_\lambda^{(+)}}{(1+gD_\gamma)g\sum_{\lambda\neq\gamma}J_\lambda^{(+)}+1}=\frac{2\pi^2gC_\gamma\sum_{\lambda:\text{open}}p_\lambda|v_\lambda(p_\lambda)|^2}{[|(1+gD_\gamma)g\sum_{\lambda\neq\gamma}J_\lambda^{(+)}+1|]^2},$$

where $p_\lambda^2+\varepsilon_\lambda=E$.

This suggests the definitions of the total width Γ^γ and the partial widths Γ_λ^γ:

$$\Gamma^\gamma=\sum_{\lambda:\text{open}}\Gamma_\lambda^\gamma, \tag{3.25}$$

where the summation is over the open channels, and

$$\Gamma_\lambda^\gamma=\frac{4\pi^2gC_\gamma p_\lambda|v_\lambda(p_\lambda)|^2}{[|(1+gD_\gamma)g\sum_{\lambda\neq\gamma}J_\lambda+1|]^2}. \tag{3.26}$$

Eq. (3.24) now reads

$$\mathcal{C}_{\beta\alpha}(E)=\mathcal{C}_{\beta\alpha}^{(p)}-\exp[2i\xi']\frac{\frac{1}{2}\sqrt{\Gamma_\beta^\gamma\Gamma_\alpha^\gamma}}{E-E_\gamma+i(\Gamma^\gamma/2)}, \tag{3.27}$$

where

$$\xi'=\xi+\tfrac{1}{2}\arg[v_\beta(p_\beta)v_\alpha^*(p_\alpha)] \tag{3.28}$$

and

$$E_\gamma=\mathcal{E}_\gamma+\operatorname{Re}\frac{g^2C_\gamma\sum_{\lambda\neq\gamma}J_\lambda^{(+)}(E)}{(1+gD_\gamma)g\sum_{\lambda\neq\gamma}J_\lambda^{(+)}(E)+1}. \tag{3.29}$$

$\mathcal{C}_{\beta\alpha}^{(p)}$ corresponds to the potential scattering amplitude $\exp[i\delta_p]\sin\delta_p$ of elastic scattering and may be termed the direct interaction amplitude. For the discussion of the physical significance of $\mathcal{C}_{\beta\alpha}^{(p)}$, the reader is again referred to Feshbach's paper.

When the resonance term predominates over the direct interaction amplitude, Eq. (3.29) gives

$$\sigma_{\beta\alpha}(E)=\frac{4\pi}{p_\alpha^2}|\mathcal{C}_{\beta\alpha}|^2\simeq\frac{\pi}{p_\alpha^2}\frac{\Gamma_\beta^\gamma\Gamma_\alpha^\gamma}{(E-E_\gamma)^2+(\Gamma^\gamma/2)^2}, \tag{3.30}$$

which, of course, agrees with the conventional result [13].

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[13] J. M. Blatt and V. F. Weisskopf: *Theoretical Nuclear Physics* (New York, 1952), p. 558.

4. – Concluding remarks.

First of all in this paper, we have constructed a model which is capable of describing inelastic scattering, for which the Low equation is exactly soluble. Our model is characterized by its close resemblance to the many channel system in ordinary quantum mechanics. Therefore we expect some general conclusions drawn from the study of this model remain valid in more realistic models. We note that the Hamiltonian (2.6)–(2.8) can be used in discussing an $(N+1)$ particle system, with the proviso made by Feshbach, although the meaning of the potentials $V_{\mu\lambda}$ should be slightly modified to include the particle identity and associated effects. With the somewhat unnatural assumption that the potentials so reinterpreted are separable, our results still hold.

We have derived the branching ratio theorem (2.24) and the cross-section theorem (2.28) from the Low equation. This suggests the possibility of utilizing the Low equation in studying many channel processes of more realistic nature even if it may not be soluble. Our derivation of the Breit-Wigner formula is based on the general properties of the generalized R-function. Through our derivation, we have made it plausible that the Breit-Wigner formula presumably applies to a wide class of theories—local or non-local—contrary to the context in which it is *usually* derived. What we have done in the derivation is essentially to explicate the « hidden structures » of, not the unperturbed Hamiltonian, but the « diagonal » Hamiltonian (3.2.).

Our last remark is relevant to the case when the interaction is not separable, *i.e.*, when the interaction is not expressible by a one-term dyad ([14]) of the form of Eq. (2.11). If one can in some way express the interaction Hamiltonian as sum of an operator and a one-term dyad, then one can treat the problem by the Chew-Low formalism for two interactions ([15]). Such a separation is indeed possible and can be carried out most lucidly by the complex potential formalism ([5]) of Feshbach. Moreover, the « hidden structure » responsible for the resonances becomes explicit automatically in this separation. The rest of this Section will be devoted to the expansion of this remark.

Let Y be the part of the interaction Hamiltonian (2.8) which connects the open channels to the closed channels, X the rest,

$$H_I = X + Y \,. \tag{4.1}$$

Here we do not specify the details of the interaction; the following discussion applies to the local potentials as well as to the non-local potentials. The

([14]) B. FRIEDMAN: *Principles and Techniques of Applied Mathematics* (New York, 1956), p. 28 *a. foll.*

([15]) S. D. DRELL and F. ZACHARIASEN: *Phys. Rev.*, **105**, 1407 (1957).

scattering amplitude is

$$T_{\beta\alpha} = (\Psi_\beta^{(-)}, H_I\psi_\alpha) = (\Psi_\beta^{(-)} X\psi_\alpha) + (\Psi_\beta^{(-)}, Y\psi_\alpha)\,, \tag{4.2}$$

where

$$\Psi_\beta^{(-)} = \psi_\beta + \frac{1}{E - i\eta - H_0}[X+Y]\Psi_\beta^{(-)}\,. \tag{4.3}$$

The subscripts α and β denote the incident channel, the energy of the system and any other quantum numbers which specify the system completely. By the standard technique [16] Eq. (4.2) can be written as

$$T_{\beta\alpha} = (\Psi_\beta^{(-)}, [X+Y]\psi_\alpha) = (\chi_\beta^{(-)}, X\psi_\alpha) + (\Psi_\beta^{(-)}, Y\chi_\alpha^{(+)})\,,$$

where

$$\chi_\lambda^{(\pm)} = \psi_\lambda + \frac{1}{E_\lambda \pm i\eta - H_0} X\chi_\lambda^{(\pm)}\,. \tag{4.5}$$

Using the integral equation

$$\Psi_\beta^{(-)} = \chi_\beta^{(-)} + \frac{1}{E_\beta - i\eta - H_0 - X} Y\Psi_\beta^{(-)}\,, \tag{4.6}$$

we re-express $T_{\beta\alpha}$ of Eq. (4.4):

$$T_{\beta\alpha} = \left(\chi_\beta^{(-)}, X\chi_\alpha) + (\Psi_\beta^{(-)}, Y\frac{1}{E_\beta + i\eta - H_0 - X} Y\chi_\alpha^{(+)}\right). \tag{4.7}$$

In the above derivation we use the fact that

$$(\chi_\beta^{(-)}, Y\chi_\alpha^{(+)}) = 0\,.$$

For Y is assumed to connect the open channels to the closed channels, and both the α- and β-channels are tacitly assumed to be open. Comparing Eqs. (4.7) and (4.4), we find that

$$T_{\alpha\beta} = (\Psi_\beta^{(-)}, [X+Y]\psi_\alpha) = \left(\Psi_\beta^{(-)}, \left[X + Y\frac{1}{E_\beta + i\eta - H_0 - X} Y\right]\psi_\alpha\right). \tag{4.8}$$

Eq. (4.8) leads us to define the complex interaction Hamiltonian H_c by

$$H_c(E) \equiv X + Y\frac{1}{E + i\eta - H_0 - X} Y\,, \tag{4.9}$$

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[16] M. GELL-MANN and M. L. GOLDBERGER: *Phys. Rev.*, **91**, 398 (1953). Eq. (4.4) of that paper contains a trivial error.

in analogy to the complex potential. The complex interaction Hamiltonian H_c is not in general hermitian. It does nonetheless give an equivalent description of the scattering process in the sense of Eq. (4.8). $H_c(E)$ may be written in terms of χ_λ as

$$H_c(E) = X + \sum_\lambda Y\chi_\lambda^{(\ )})(E + i\eta - E_\lambda)^{-1}(\chi_\lambda^{(\ )}, Y\,, \tag{4.10}$$

where the summation is over the channels, energies, and other quantum numvers. (The summation is to be replaced by integration for the scattering states and either the outgoing state vectors $\chi_\lambda^{(+)}$ or the incoming state vectors $\chi_\lambda^{(-)}$ may be used for $\chi_\lambda^{(\ )}$). The hidden structure of the Hamiltonian $H_0 + X$ is made explicit through the energy denominator on the right of Eq. (4.10). Suppose now the energy of the system is close to the energy E_r of the bound state χ_r. Then we write $H_c(E)$ as

$$H_c(E) = U_r + \frac{1}{(E - E_r)^{\frac{1}{2}}} Y\chi_r)(\chi_r Y \frac{1}{(E - E_r)^{\frac{1}{2}}}\,. \tag{4.11}$$

and the desired decomposition of the interaction Hamiltonian is effected.

RIASSUNTO (*)

Si esamina l'equazione di Low per un processo di scattering non relativistico servendosi di un modello in cui la sorgente di scattering può esistere in più di uno stato, e l'interazione « media » tra la sorgente e la particella è fattorizzabile. Il teorema della branching ratio e il teorema ottico si derivano direttamente dall'equazione di Low. Si risolve l'equazione di Low col metodo di Castillejo, Dalitz e Dyson e si determina la soluzione unica esaminando lo spettro dell'hamiltoniana imperturbata. Si studiano i fenomeni di risonanza e la formula della risonanza di Breit-Wigner si deriva utilizzando le proprietà della funzione R generalizzata. Si indica la più generale applicabilità del metodo.

(*) *Traduzione a cura della Redazione.*

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Energy Distribution of External Bremsstrahlung from β-rays of ^{90}Y.

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(ricevuto il 15 Maggio 1959)

Summary. — The external bremsstrahlung spectrum from β-rays of ^{90}Y stopped in Al, Ag and Pb is investigated between 0.4 and 2 MeV. The experimental spectra are compared with the theoretical ones, calculated according to the Bethe-Heitler formula for external bremsstrahlung cross-section. A disagreement (535% for Pb at 2 MeV) increasing with energy and Z is shown.

1. – Introduction.

The external bremsstrahlung (EB) from several β-ray emitters has been studied experimentally by various investigators; their measurements have shown that the bremsstrahlung spectra have a shape roughly independent of the atomic number Z of the β-absorber and a large intensity of low-energy quanta. As far as concerns the energy distribution of EB produced by β-ray emitters, whose maximum energy is a few mc^2 units towards the high energy limit of the spectrum, reliable measurements are at present lacking, with the exception of a detailed investigation by Lidén and Starfelt [1] on EB from β-rays of ^{32}P ($E_{max} = 1.71$ MeV): for light elements the observed spectra were found to agree substantially with the Bethe-Heitler [3] EB production cross-section formula, but for heavy elements and high energies they considerably exceeded the predicted values. A disagreement between experimental results and theoretical previsions [2] increasing with energy and Z is however not

[1] K. Lidén and N. Starfelt: *Phys. Rev.*, **97**, 419 (1955).

[2] H. Bethe and E. Salpeter: *Handb. d. Phys.*, vol. **35**, edited by S. Flügge (Berlin, 1957), p. 409.

[3] H. Bethe and W. Heitler: *Proc. Roy. Soc.* (*London*), A **146**, 83 (1934).

surprising: in fact the Bethe-Heitler ([3]) cross-sections are based on Born's approximation which breaks down with large Z and in the vicinity of the high energy limit of the spectrum. Unfortunately no calculations of EB production cross-section without Born's approximation have been made for bremsstrahlung radiation with energy between 10 keV and 20 MeV, though SOMMERFELD ([4]) (for energy less than 10 keV) and BETHE and MAXIMON (for energy more than 20 MeV), have given formulae which lead to calculated spectra in agreement with experiments. It may be observed, however, that ELWERT ([5]) gave semiempirically a correction factor to the Bethe-Heitler formula, but this factor seems, according to BETHE and SALPETER ([2]), to be incorrect.

New interest in EB accompanying β-rays was aroused when it was discovered that electrons from β decay are longitudinally polarized ([6,7]) and that, consequently, the forward bremsstrahlung emitted is circularly polarized ([8]). In particular, the circular polarization of EB from pure β-ray emitters was studied in order both to obtain information about the influence of Z on it and to check the efficiency of γ-polarimeters. In this connection it is emphasized that the relation between circular polarization and energy is affected by EB production cross-sections to an extent which is sensitive to the deviations from Bethe-Heitler formula.

In the present paper the results of an investigation on the spectral shape of the external bremsstrahlung produced by β-rays of ^{90}Y ($E_{max} = 2.28$ MeV) in various absorbers are reported.

2. – Experimental method.

The source of ^{90}Sr (40 mC) in equilibrium with ^{90}Y used, was made by a disc of silver 0.05 mm thick containing uniformly dispersed strontium carbonate, protected by a facing layer of silver 0.1 mm thick and backed by 0.8 mm of silver. The β-ray spectrum was therefore formed by the superposition of the spectrum of the ^{90}Sr (with a $E_{max} = 0.54$ MeV) and that of ^{90}Y (with an E_{max} of 2.28 MeV); as the lower limit in this investigation was at 0.4 MeV, only ^{90}Y gave contribution to the observed γ-spectrum.

The source was placed at a distance of 50 cm from the γ-detector, and a lead absorber was placed half-way between the γ-ray emitter and the detector

([4]) A. SOMMERFELD: *Ann. d. Phys.*, **11**, 257 (1931).
([5]) G. ELWERT: *Ann. d. Phys.*, **34**, 178 (1939).
([6]) T. D. LEE and C. N. YANG: *Phys. Rev.*, **104**, 254 (1956).
([7]) H. FRAUENFELDER, R. BOBONE, E. VON GOELER, N. LEVINE, H. R. LEWIS, R. N. PEACOCK, A. ROSSI and G. DE PASQUALI: *Phys. Rev.*, **106**, 386 (1957).
([8]) R. L. GLÜCKSTERN, M. H. HULL jr. and G. BREIT: *Phys. Rev.*, **90**, 1026 (1953).

so as to reduce the total number of pulses. A NaI(Tl) crystal (1.5 in. diameter, 1 in. thick), optically coupled to the photocatode of a Du Mont 6292 photomultiplier was used and the γ-spectrum was investigated with a fast ten-channel pulse-analyser ([9,10]).

The channel width was of 80 keV and the whole spectrum was measured within a period of 30 min at a time, for a total period of (8÷10) hours. After every measurement energy calibration was checked with the following monochromatic radiations: 835 keV from ^{54}Mn, 511 and 1277 keV from ^{22}Na and the line given by the sum of ^{22}Na 511 and 1277 keV lines.

The ratio between the background and observed pulse height distribution was about 0.5 at the upper limit of the γ-spectrum, but it decreased rapidly with decreasing energy. It is to be noted that the background was measured by interposing a very thick lead absorber midway between the source and the crystal. The size of this absorber was chosen so that it intercepted only the direct beam without perturbing the scattered radiation, of which thus account was taken.

3. – Results.

The observed pulse height distribution (from which previously background was subtracted) must be corrected before being compared with theoretical evaluation.

The following corrections ([11]) were considered:

a) Resolving power of the spectrometer. For this correction use was made of the relation ([12]):

$$N_{\text{corr}} = N_{\text{uncorr}} - \frac{E^2\eta^2}{11.088}\frac{\mathrm{d}^2 N_{\text{uncorr}}}{\mathrm{d}E^2}, \tag{1}$$

where E is the energy corresponding to the point of the spectrum to be corrected, N_{corr} and N_{uccorr} are respectively the intensities of the corrected and uncorrected spectra, and η is the half-width of a peak corresponding to a monochromatic line of energy E observed with the detection chain. The values of the half-widths used were taken from the curve E vs. η given by Bisi and Zappa ([13]) (η at three different energies was measured and found in agreement within a few percent with the values given by Bisi and Zappa).

([9]) E. Gatti: *Nuovo Cimento*, **11**, 153 (1954).
([10]) S. Colombo, C. Cottini and E. Gatti: *Nuovo Cimento*, **5**, 748 (1957).
([11]) K. Lidén and N. Starfelt: *Ark. f. Fys.*, **7**, 427 (1954).
([12]) W. R. Dixon and J. H. Aitken: *Can. Journ. Phys.*, **36**, 1624 (1958).
([13]) A. Bisi and L. Zappa: *Nucl. Instr.*, **3**, 17 (1958).

b) Compton electron distribution. When subtracting the Compton electron distribution, a flat Compton pulse height distribution was assumed, and the calculation made use of the peak to total ratio as given by BELL ([14]): this ratio was also experimentally determined, and the measured values were in agreement, within a few percent with Bell's.

c) Other corrections. The experimental spectra were also corrected for the γ-efficiency of the crystal, absorption in the lead disc, and self absorption (only for lead) of the γ-quanta in the β-ray stopper: this last correction was made, considering as though all photons crossed an absorber half as thick as the stopper. No correction was made for backscattering and X-rays escape (negligible above 0.4 MeV), and for dead time of the chain and pile-up of pulses owing to the lead absorber and the distance between source and detector.

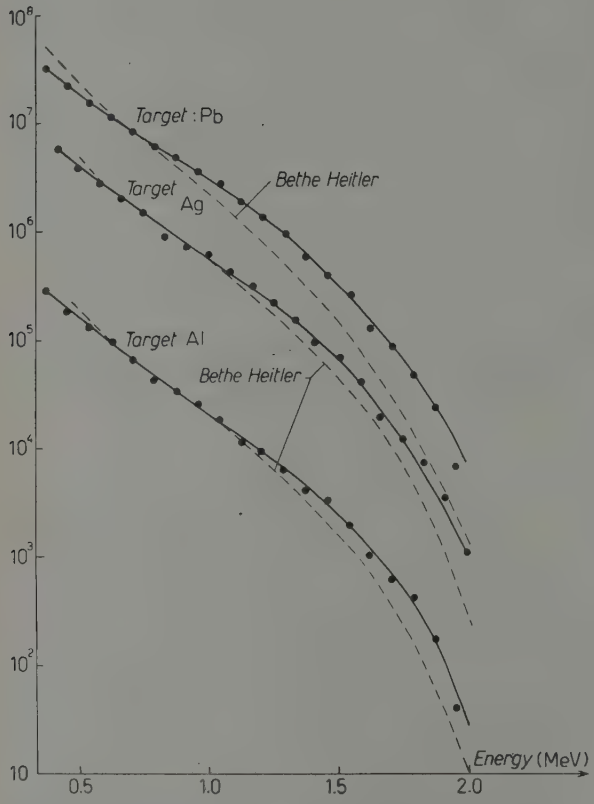

Fig. 1. – Comparison of ^{90}Y external bremsstrahlung spectra for Al, Ag and Pb with theoretical ones. The solid lines represent theoretical spectra calculated making use of Bethe-Heitler cross-sections. The statistical error is smaller than the size of the points.

Finally by dividing by the theoretically calculated ratio of the sum of external and internal bremsstrahlung, to external bremsstrahlung, the EB spectra were obtained. It is interesting to observe that, owing to the rapid decrease of the intensity of spectra with the increase of energy, the whole correction was only a few percent of the measured values. The EB spectra are reported in Fig. 1 where they are compared with theoretical curves.

It is to be noted that

([14]) P. R. BELL: *Beta and Gamma Ray Spectroscopy*, edited by K. SIEGBAHN (Amsterdam, 1955), p. 139.

(Section 2), as the source was covered with silver (0.1 mm), the curves indicated in Fig. 1 by Al (or Pb) should, to be exact, be regarded as a superposition of EB in Ag, and of EB in Al (or Pb).

In this connection it must be noted that, as the EB production cross-sections increase with Z^2 and the disagreement between theory and experiment increases with Z, the spectra for materials with Z larger than the Ag's one are more reliable than the ones with lower Z. In fact for the formers the error due to the superposition of spectra is percentually lower and moreover acts so to diminish the disagreement. The contrary happens for the latters. No correction was made for this source of error as it would have been too approximate.

Thereof, while the measurements for Ag are quite correct, those for Pb are not so good, but, as from a rough calculation the point at 2 MeV may be affected (as a maximum) with an error of 50% while the measured disagreement between experiment and theory is 535%, they are quite satisfactory for our purposes. A measurement was also made with a target of Al ($Z = 13$). For the above mentioned reasons the disagreement shown in Fig. 1 may be attributed to the silver of the source, however, as the effect of silver decreases with energy, it may be interesting to observe that the experimental and theoretical curves cross themselves with a lower angle for Al than for Ag.

Partially for this reason, and in view, above all, of the fact that the absolute intensity of the source was not known with adequate approximation, the comparison of experiment with theory was made, separately for Al, Ag and Pb, assuming arbitrarily for experimental and theoretical curves the same value at 0.7 MeV.

4. – Conclusions.

Theoretical evaluation of EB spectra from β-rays is made as follows: as the probability of emitting quanta of energy k for an electron of initial energy E^* is

$$n(k, E^*) = \int_{I+k}^{E^*} \frac{\Phi N}{-\mathrm{d}E/\mathrm{d}x}\,\mathrm{d}E\,, \tag{2}$$

where Φ is the EB production cross-section, N is the number of atoms per cm^3 of the stopping target and $-\mathrm{d}E/\mathrm{d}x$ the energy loss per cm of the electron, the resulting EB spectrum $S(k)$, when β-rays from a pure β-emitter are

stopped, is

$$S(k) = \int_{I+k}^{E_{\max}} P(E^*) n(k, E^*) \, \mathrm{d}E^* . \tag{3}$$

Eqs. (2) and (3) were numerically integrated making use of the values of the energy loss calculated according to Bethe-Bloch formula ([15]) and of EB production cross-section values derived from Bethe-Heitler formula. $P(E)$ in (3) is the distribution of β-ray spectrum of ^{90}Y, and was calculated taking into account the shape factor for unique first forbidden transition ([16-17]).

It is to be noted that the Bethe-Heitler formula, used for our theoretical evaluation, being valid in Born's approximation, breaks down toward the high energy limit of the spectrum and for large Z, giving rise to spectra which lie below the experimental ones.

In particular, the Bethe-Heitler formula gives a value of zero for the probability that an electron radiates a photon of the same energy, while on the basis of experiments this probability does not seem to be zero.

The comparison (Fig. 1) of the experiment with theory shows, as said above, a disagreement increasing with energy and Z: this is also shown in Table I

TABLE I. – *Ratio between experimental and theoretical values of* ^{90}Y *external bremsstrahlung for* Al, Ag *and* Pb *at various energies.*

Energy (MeV)	Al	Ag	Pb
0.5	0.84	0.80	0.77
0.7	1.00	1.00	1.00
1	1.00	1.04	1.36
1.5	1.50	1.50	2.15
2	2.55	3.71	5.35

where the ratio of the experimental to theoretical values at some values of energy and for three different materials is reported.

([15]) W. PAUL and H. STEINWEDEL: *Beta and Gamma Ray Spectroscopy*, edited by K. SIEGBAHN (Amsterdam, 1955), p. 9.

([16]) M. E. ROSE: *Beta and Gamma Ray Spectroscopy*, edited by K. SIEGBAHN (Amsterdam, 1955), p. 272.

([17]) M. E. ROSE, C. L. PERRY and N. M. DISMUKE: *Beta and Gamma Ray Spectroscopy*, edited by K. SIEGBAHN (Amsterdam, 1955), p. 884.

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Thanks are due to Prof. G. Bolla for his kind interest and to Prof. A. Bisi for helpful discussions.

RIASSUNTO

È stato studiato lo spettro della bremsstrahlung esterna prodotta dal frenamento dei raggi β dell'^{90}Y in assorbitori di Pb, Ag ed Al. Le curve sperimentali sono state confrontate con quelle calcolate in base alla formula di Bethe-Heitler. Si è potuto così osservare un disaccordo notevole (535% per il Pb a 2 MeV) tra le due serie di curve, disaccordo che cresce col crescere dell'energia e del numero atomico Z dell'assorbitore.

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The Diffusion of Tritium in Liquid Argon.

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(ricevuto il 20 Maggio 1959)

Summary. — A simple expression of the diffusion coefficient as function of the temperature at constant pressure for mixtures of perfect liquids previously suggested by us has been checked in the case HT-liquid Argon. The value obtained by the capillary method adopted for low temperature work is $D_{exp}(\text{HT-A}) = (3.17 \pm 0.15) 10^{-5}$ cm²/s at $T = 84.5$ °K in satisfactory agreement with the calculated value $D_{ca'c}(\text{HT-A}) = 2.94\ 10^{-5}$ cm²/s.

1. – Introduction.

In a recent investigation (1) the diffusion coefficients of A and HT in liquid Nitrogen have been measured by using the capillary method. The results were discussed in terms of the dependence of the diffusion coefficient upon the molecular parameters of the different substances and a simple expression $D = D(T, p = \text{const})$ was given for any diffusion of mixtures of two perfect liquids. In order to test this relation we have now investigated the diffusion coefficient of HT in liquid Argon at a pressure of about 1 atmosphere and at the temperature of 84.5 °K.

2. – Experimental procedure.

2·1. *Diffusion apparatus.* – The diffusion coefficient was measured by the capillary method of Anderson and Saddington (2) adapted by us for low temperature work (1). The temperature regulation for this experiment is pro-

(1) G. Cini-Castagnoli, G. Pizzella and F. P. Ricci: *Nuovo Cimento*, **10**, 300 (1958).
(2) J. S. Anderson and K. Saddington: *Journ. Chem. Soc. London*, S 381 (1949).

vided by controlling the pressure over a bath of liquid oxygen. The conversion from the oxygen vapor pressure to temperature is made by Linder tables. The temperature is kept constant in each experiment within less than 0.1 °K. The capillary is a brass tube 4/10 of mm in diameter, with two different lengths in the various runs. We wish to mention that some experiments performed with capillaries worn out at the ends for a length of about 2 mm by repeated sealing operations, were clearly turbolent. Actually in these irregular runs (which are not considered in this paper) the diffusion coefficient turned out to increase with decreasing diffusion time.

The procedure adopted to carry out an experiment is the same as the one of the earlier paper.

2·2 *Counting apparatus.* – HT concentration in the Argon diffusion samples was measured by using G.M. counters with radioactive filling, as the low energy of Tritium (18 keV) requires. The measurements need high precision and accuracy in the filling operations, in order to establish every time the same working conditions of the counters. As a matter of fact the plateau characteristics and convenience of measurements are influenced by the composition and partial pressures of the counting mixture constituents. The G.M. anode is a tungsten wire of diameter 0.025 mm, and the cathode a 4 cm diameter brass tube with a 30 cm length (the effective counting volume is about 250 cm^3). Care must be devoted particularly to the cathode surface polishing, after which the counter must be warmed with hydrogen and then evacuated with a diffusion pump for several days. The tube is permanently surrounded by a 3 cm lead shield to reduce laboratory background variations. The filling line and bank of counters are illustrated in Fig. 1. The details of the filling procedure are the following: the active sample to be analized is introduced in the volume between *a*, *b*, *c*, *d* stopcocks; its pressure (about 5 cm Hg) is then measured at the manometer *m* with a precision of $5 \cdot 10^{-2}$ mm Hg by means of a cathetometer, its temperature is taken with a 1/10 °C thermometer. The gas then flows into the counter through stopcock *c*, reducing its pressure by a constant factor known at 3‰, characteristic of the line. The partial pressure of tagged argon in the counter (about 1 mm Hg) is known in this way within 5‰. The filling mixture is then completed with Argon and Alcool in Trost proportion through *e* and *f* stopcocks at a pressure of 10 cm Hg determined on the manometer *M*. The counters above described have the following characteristics: threshold voltage ~ 1240 V, plateu of $(300 \div 400)$ V beginning 60 V above threshold with slope 2% per 100 V. Four hours after the filling counting conditions are stabilized. Counting rates were measured at 60 V intervals for the whole length of the plateau, and the activity of the sample was obtained from the middle point of the plotted plateau. The background of 120 counts/min is constant between two subsequent measurements within 1%. Following the counting

rate measurements the tube was flushed 4 times with Hydrogen and pumped out for about four hours at a pressure of less than 10^{-3} mm Hg.

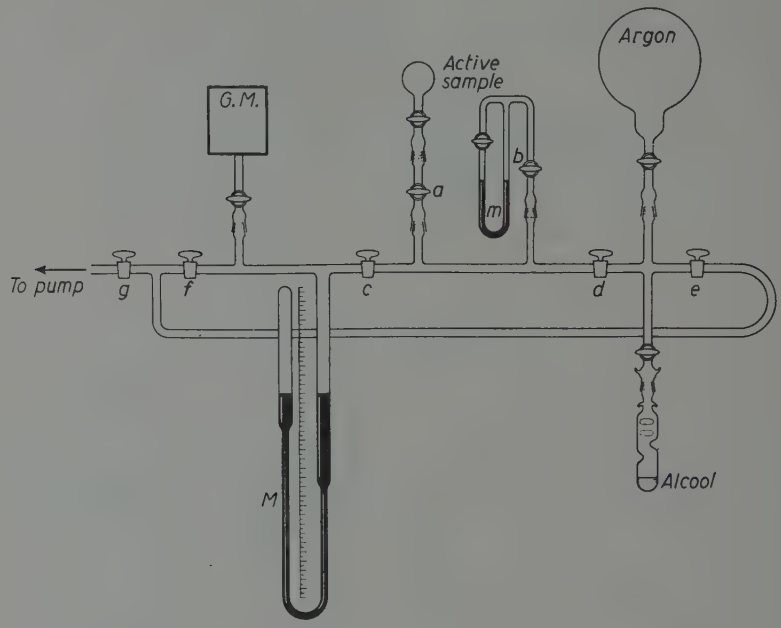

Fig. 1. – Gas lines and G. M. bank for tritium counting.

In order to check the reproducibility of the measurements four or five consecutive counter tube fillings were made using the same sample. Constancy of the counts per minute per mm Hg of tagged Argon was observed within one or maximum two percent. The results are shown in Fig. 2. The satisfactory agreement demonstrates that the procedure described above represents a convenient quantitative method for the estimation of Tritium in Argon samples. The diffusion samples were measured at least two times

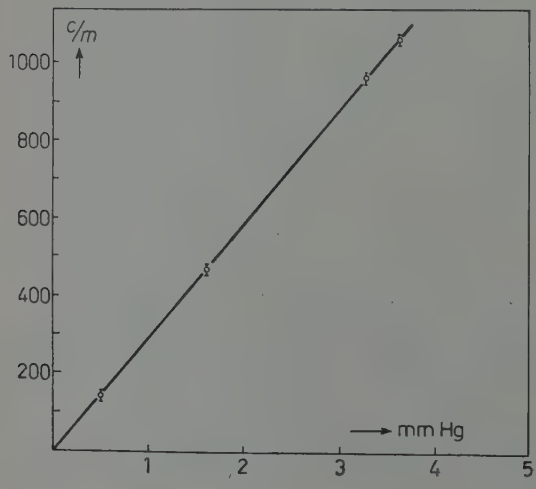

Fig. 2. – Counting rate vs. partial pressure of tritium sample.

and their activities varied in the range (100 ÷ 2000) counts/min per mm Hg in the counter.

The constancy of the counting efficiency was frequently checked by inserting a ^{60}Co salt source in a well defined position outside the counter tube.

3. – Experimental results and discussion.

The diffusion coefficient D is calculated by the well known formula:

$$\frac{C_s - C_{d_{Av}}}{C_s - C_i} = \frac{8}{\pi^2} \sum_{n=0}^{\infty} \left\{ \frac{1}{(2n+1)^2} \exp\left[- \frac{(2n+1)^2}{4L^2} \pi^2 \Delta t \right] \right\},$$

where C_s is the concentration of tracer in the diffusion chamber, $C_{d_{Av}}$ is the average concentration in the capillary after time t, c_i is the initial concentration of tracer in the capillary at zero time, L is the length of the capillary.

The diffusion coefficient of HT in liquid Argon from a 10^{-7} % HT-A mixture was studied at the temperature of 84.5 °K and at a pressure of about 70 cm Hg. The results of the measurements are listed in Table I. The corresponding errors were calculated by taking into account those due to the concentration measurements and to the length of the capillary.

TABLE I. – *Experimental results for* HT-A *diffusion.*

Run no.	Length of the capillary (cm)	Time (s)	Temperature (°K)	D (cm²/s·10⁵)
1	1.520±0.020	9960	84.54±0.05	3.10±0.20
2	1.520±0.020	7560	84.54±0.05	3.40±0.40
3	2.295±0.005	5400	84.55±0.05	3.15±0.17
4	2.295±0.005	11160	84.51±0.05	3.20±0.20
			$T = 84.53$	$D = 3.17 \pm 0.15$

The weighted mean of the obtained values is:

$$D(\text{HT-A}) = (3.17 \pm 0.15) \cdot 10^{-5} \text{ cm}^2/\text{s} .$$

This experimental value of D(HT-A) can be compared with the value calculated by means of the expression for the diffusion of mixtures of perfect liquids given in the earlier paper [1]

$$(1) \qquad D = 9 \cdot 10^3 \frac{\sigma_1^2}{\sigma_{i1}} \sqrt{\frac{\varepsilon_{i1}}{m_{i1}}} \exp\left[- \frac{2.6 \varepsilon_1}{kT} \right] \text{ cm}^2/\text{s} ,$$

we obtain

$$D(\mathrm{HT\text{-}A})_{\mathrm{calc}} = 2.94 \cdot 10^{-5} \ \mathrm{cm^2/s} \ .$$

The agreement is quite satisfactory.

The diffusion mechanism underlying equation (1) seems therefore to be confirmed by this further measurement. However, beefore discussing eq. (1) in detail, it seems necessary to make clear the reasons of the discrepancy already ([1]) mentioned between the experimental value of Argon self diffusion and the value calculated with eq. (1). Measurements on the ^{37}A-^{40}A diffusion are therefore in progress. We only point out here that expression (1) of the diffusion coefficient shows that the diffusion process proceeds by subsequent steps involving only pairs of molecules.

RIASSUNTO

Una semplice espressione $D = D(T, p = \mathrm{cost})$ da noi data per la determinazione del coefficiente di diffusione di miscele binarie di liquidi perfetti viene qui controllata nel caso Tritio-Argon liquido. La misura eseguita con il metodo del capillare adattato alla bassa temperatura fornisce il risultato $D = (3.17 \pm 0.15) \cdot 10^{-5}$ cm^2/s in buon accordo con il valore previsto $D = 2.94 \cdot 10^{-5}$ cm^2/s.

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Further Developments of the Covariant Functional Formalism for Interacting Fields. - III.

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(ricevuto il 22 Maggio 1959)

Summary. — In this work the covariant functional formalism developped in two previous articles is extended to the case of coupled boson fields and of fermion fields, either free or coupled to boson fields. The sufficiency of the knowledge of the energy in order to determine the evolution of coupled fields is discussed, and some properties of the state vector, resulting immediately from the covariant functional formulation in the Schrödinger representation, are given. It is shown that at the classical limit ($\hbar \to 0$), the solution of the equations of motion for coupled fields exists and is unique.

1. – Introduction.

In two previous works [1] we have developped a covariant Hamiltonian formalism for boson fields, free or interacting with known current distributions, passing from the classical to the quantum representation by means of a covariant functional formalism.

Some precautions are necessary in extending these results to the case of fermion fields. In II we indicated the works in which the Hamiltonian formalism is treated, recalling the criteria for the integration of the classical canonical equations. These methods are not directly applicable to fermion fields, since they refer to systems of second-order hyperbolic differential equations, such as the classical equations for the theory of boson fields; and we

(1) R. S. Liotta: *Nuovo Cimento*: **3**, 438 (1956); **8**, 798 (1958), referred to as I and II.

note that in the deduction of the methods of integration, the fact that the boson fields fulfil commutation relations plays an essential role.

The situation is different in the case of fermion fields. The Dirac equations describing the motion of the electron and of the positron are first-order differential equations, and therefore their integration is possible when the functions $\psi(x)$, $\bar{\psi}(x)$, which assume the roles of Lagrange co-ordinates for the electron and positron respectively, are known on a given space-like surface σ_0. Furthermore, the spur of the canonic tensor is proportional to the mass of the electron and is identically zero, unless the anti-commutation properties are explicitly introduced at the beginning. The latter means only the changing of some formal properties which can be coherently developed [2]; whereas the former fact—the proportionality of the spur of the canonic tensor to the mass of the fermion—implies that if the neutrino has mass zero, in this scheme (*) it must obey an equation of the second order as has recently been proposed by FEYNMAN and GELL-MANN [3].

These considerations suggest making a more profound analysis in the $T_{\mu\mu}$ scheme of the classical and quantum mechanical aspects of the spinor equations.

The solution of the equations of motion of field theory for free fields and for fields in the presence of given current distributions is well known, but little is known about the existence and uniqueness of the solution for coupled fields. Recently there has been some speculation [4] as to whether a field theory could be constructed satisfying various fundamental postulates, such as causality, the possibility of accounting for physical particles (for which renormalization processes should not be necessary), without having to postulate an equation of motion.

Our purpose is to try to show that, at least in certain cases, an equation of motion for coupled fields can be written for which,under certain hypotheses, the solution exists.

Such a theory must obviously be covariant and renormalizable. Leaving apart for the moment the problem of renormalization, we will show that: for coupled boson fields (electromagnetic field interacting with meson field) and for boson fields coupled with fermion fields (electromagnetic field interacting

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(*) From now on we will call the classical covariant Hamiltonian formalism the « $T_{\mu\mu}$ scheme », and the quantum-mechanical, « $\mathcal{C}_{\mu\mu}$ scheme », while continuing to use the symbols $H = T_{\mu\mu}$ and $\mathcal{H} = \mathcal{C}_{\mu\mu}$ in the formulae. Analogously, we will call the usual Hamiltonian formalisms the « T_{44} scheme » (classical) and « $\mathcal{C}_{44}$ scheme » (quantum-mechanical).

(2) E. MAJORANA: *Nuovo Cimento*, **5**, 171 (1937).

(3) R. P. FEYMANN and M. GELL-MANN: *Phys. Rev.*, **109**, 193 (1958).

(4) H. LEHMANN, K. SYMANZIK and W. ZIMMERMANN: *Nuovo Cimento*, **2**, 425 (1958), and later works.

with Dirac field), sufficient conditions for the existence of the solution at the classical limit can be established.

However, this program poses a preliminary problem: if the equation of Tomonaga-Schwinger as usually constructed, really is the equation of motion for field theory.

In the next paragraphs we will make a preliminary study of this problem, referring to several physically interesting cases of free and interacting fields, and showing how, without postulating *a priori* a Schrödinger equation for fields, it is possible, by means of the special theory of relativity and the correspondence principle, to construct an equation of motion which coincides in various cases with the Tomonaga-Schwinger equation.

2. - Construction of the equation of motion in field theory.

The procedure leading to the formulation of the Tomonaga-Schwinger equation in field theory is well known. First of all let us consider an electron-positron-photon system. The total energy density is written in the form $H^{(e)}+H^{(\gamma)}+H_i=T_{44}^{(e)}+T_{44}^{(\gamma)}+H_i$; $H_i \doteq J_\mu A_\mu$ and it is postulated that the equation of motion be the time-dependent Schrödinger equation. Since $H^{(e)}$ and $H^{(\gamma)}$ the energies of the free electron and photon fields, are not covariant, while H_i is covariant, a canonical transformation eliminates the non-covariant part from the Hamiltonian so that the evolution of the system is governed by the covariant part alone. It is therefore sufficient to define the state vector on a family of surfaces to make the equation of motion assume the form

$$i\hbar \frac{\delta \Phi}{\delta \sigma(x)} = \mathcal{H}_i(x)\Phi \,, \tag{1}$$

which is evidently covariant.

In the case of a pseudoscalar meson field φ interacting with an electromagnetic field ([5]), the Hamiltonian of interaction T_{44} contains the derivatives $\partial\varphi/\partial x_k$ and is not an invariant. Nevertheless ([6]), imposing the necessary condition for integrability—that is, imposing the physical requirement that on all points of a spatial surface the fields be measurable ([7])—two additive terms

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([5]) G. Wentzel: *Quantum Theory of Fields* (1949).

([6]) S. Kanezawa and S. I. Tomonaga: *Progr. Theor. Phys.*, **3**, 1 (1948), see also Y. Miyamoto: *Progr. Theor. Phys.*, **3**, 124 (1948).

([7]) For the physical significance of such a condition see: N. Bohr and L. Rosenfeld: *Phys. Rev.*, **78**, 795 (1950); E. C. G. Stückelberg: *Phys. Rev.*, **81**, 130 (1951); E. Corinaldesi: *Suppl. Nuovo Cimento*, **10**, 2, 83 (1953); E. C. G. Stückelberg, and T. A. Green: *Helv. Phys. Acta*, **24**, 153 (1951).

to the Hamiltonian are obtained: one renders covariant the density of the interaction energy, and the other, also covariant, depends on the family of space-like surfaces considered and in the case under consideration is of the type

$$(n_\alpha A_\alpha)^2 \varphi^* \varphi \,. \tag{2}$$

TOMONAGA and KANESAWA do not discuss the presence of this term which they found. In the classical scheme (9), if, considering the entire Hamiltonian including the term (2), we were to effect a canonical transformation leaving as Hamiltonian only the term (2), we would, find an equation of motion in which the state of the system on the surface σ would depend not only on the state on the preceding surface σ_0, but also on the family of surfaces chosen to study the motion: this is clearly devoid of physical significance unless we introduce some approximations which we will see later.

On the other hand, in the quantum mechanical scheme, MATTEWS (8) has shown that in the calculation of matrix elements by the perturbation method, there is no contribution from terms dependent on the family of surfaces up to the second order. Nor, we should presume, may such terms contribute to higher-order perturbation terms. If this is always true, we should conclude that such terms must necessarily be present in the equation of motion but must not contribute to determine the evolution of the system.

This unsatisfactory result comes from the fact that we have postulated *a priori* that the energy determine the dynamic evolution of the system, even for systems of infinite degrees of freedom, and therefore the interaction representation is used in the above procedure. Serious objections have recently (9) been raised as to the real existence of such a representation; here, however, we wish to point out the consequences of returning from the Tomonaga-Kanezawa formulation, through the correspondence principle, to the classical formulation.

Besides, it should be noted that in the usual formulations of field theory the correspondence principle is verified by means of Ehrenfest's theorem—that is, demonstrating that the mean values of quantum-mechanical quantities behave as their classical equivalents,—while nothing is said as to the point-for-point correspondence in space-time.

These inconveniences may be eliminated by following a procedure already used in I and II and which we discuss here more amply to see its physical content. The procedure is quite analogous to that used to introduce the quantum mechanics of sistems having a finite number of degrees of freedom and

(8) P. T. MATTEWS: *Phys. Rev.*, **76**, 1658 (1949).

(9) R. HAAG: *Dan. Mat. Phys. Medd.*, **29**, 12 (1955); D. HALL and A. S. WIGHTMAN: *Dan. Mat. Phys. Medd.*, **31**, 5 (1957).

eliminates the implicit postulate that the energy determine the evolution of the fields. It makes recourse only to the principles of special relativity and to the correspondence principle, generalized for systems of infinite degrees of freedom, and permits, by means of the generalization of the W.K.B. method, the study of the point-for-point correspondence in space-time between classical and quantum mechanics. In this scheme it is therefore necessary that it be legitimate to pass from a covariant Lagrangian formulation to a covariant Hamiltonian formulation; that is, it is necessary to have a system of equations in normal form which is clearly covariant (I.10). Thus by some formal substitutions (II.18) the function $H = T_{\mu\mu}$ becomes an operator and the Tomonaga-Schwinger equation is postulated in the usual form (II.21).

With this procedure we obtain an equation of motion in the Schrödinger representation satisfying the necessary condition of integrability which does not contain terms of the type (2).

3. - Pseudoscalar meson field interacting with an electromagnetic field.

Let us consider briefly the pseudoscalar field in the absence of an electromagnetic field. From the Lagrangian ([9]):

$$L^{(\pi)} = -\frac{\partial\varphi^*}{\partial x_\nu}\frac{\partial\varphi}{\partial x_\nu} - \mu^2\varphi^*\varphi\,, \tag{3}$$

considering as Lagrangian variables φ and φ^*, by the definition (I.3), the conjugate momenta are respectively:

$$\pi_\nu = -4\frac{\partial\varphi^*}{\partial x_\nu}\,, \qquad \pi_\nu^* = -4\frac{\partial\varphi}{\partial x_\nu} \tag{4}$$

while from (I.6) the spur of the canonical tensor is:

$$H^{(\pi)} = T_{\mu\mu}^{(\pi)} = -\tfrac{1}{4}\pi_\nu^*\pi_\nu + 4\mu^2\varphi^*\varphi\,. \tag{5}$$

It is immediately verified that the covariant Hamiltonian equations (I.9) obtained from (4) and (5) are equivalent to the Klein-Gordon equation for the field considered.

For the pseudoscalar meson field in interaction with the electromagnetic field we consider the Lagrangian:

$$L = -\left(\frac{\partial\varphi^*}{\partial x_\nu} + ieA_\nu\varphi^*\right)\left(\frac{\partial\varphi}{\partial x_\nu} - ieA_\nu\varphi\right) - \mu^2\varphi^*\varphi - \frac{1}{2}\frac{\partial A_\mu}{\partial x_\nu}\frac{\partial A_\mu}{\partial x_\nu}\,, \tag{6}$$

where e is the electric charge of the meson. The momenta conjugate to the Lagrangian variables A_μ, φ, and φ^* are:

$$(7)\qquad \begin{cases} P_{\mu\nu} = -4\dfrac{\partial A_\mu}{\partial x_\nu}, \\ P_\nu = -4\left(\dfrac{\partial \varphi^*}{\partial x_\nu} + ieA_\nu\varphi^*\right), \\ P_\nu^* = -4\left(\dfrac{\partial \varphi}{\partial x_\nu} - ieA_\nu\varphi\right). \end{cases}$$

From (I.6) the spur of the canonical tensor is:

$$(8)\qquad H(P_{\mu\nu}, P_\nu^*, P_\nu, A_\mu, \varphi^*, \varphi) = -P_{\mu\nu}P_{\mu\nu} - P_\nu^*P_\nu + ieA_\nu(P_\nu\varphi - P_\nu^*\varphi) + 4\mu^2\varphi^*\varphi\,.$$

With this Hamiltonian, the covariant Hamilton equations give the Klein: Gordon equations for pseudoscalar mesons in the presence of an electromagnetifield, and the electromagnetic field equations in the presence of the currentc

$$(9)\qquad J_\mu = ie\left(\frac{\partial \varphi^*}{\partial x_\mu}\varphi - \frac{\partial \varphi}{\partial x_\mu}\varphi^*\right) - 2e^2A_\mu\varphi^*\varphi\,.$$

It is worth noting that although the Hamiltonian (8) is not invariant under a gauge transformation

$$(10)\qquad \varphi \to \varphi\exp[iex]\,,\qquad \varphi^* \to \varphi^*\exp[-iex]\,,\qquad A_\nu \to A_\nu + \frac{\partial \chi}{\partial x_\nu}\,,\quad \Box\chi = 0,$$

nevertheless classically this causes no difficulty, since the equations of motion do result gauge invariant. That is, classically the evolution of the fields is determined by the sum of the spurs of the canonical tensors of the free fields and of the spur of their interactions, since the system of the equations of motion is given by (I.9). It is easily verified that all of the classical and quantum-mechanical results contained in I and II may be extended to the case of two coupled boson fields).

Comparing the interaction part of (8):

$$(11)\qquad H^{(\pi,\gamma)} = ieA_\nu(P_\nu\varphi - P_\nu^*\varphi^*) = -4ieA_\nu\left(\frac{\partial \varphi^*}{\partial x_\nu}\varphi - \frac{\partial \varphi}{\partial x_\nu}\varphi^*\right) + 8e^2A_\nu A_\nu\varphi^*\varphi\,,$$

with the expression (3.11) given by KANEZAWA and TOMONAGA ([6]), we see that they coincide apart from the above-mentioned term (2) and an unes-

sential difference of unity. It is useful for the considerations which will follow to examine the classical significance of this term.

Of the equations (I.9) constructed with the Hamiltonian (5), those which define the conjugate momenta

$$\frac{\partial\varphi}{\partial x_\nu} = \frac{\partial H^{(\pi)}}{\partial \pi_\nu}, \qquad \frac{\partial\varphi^*}{\partial x_\nu} = \frac{\partial H}{\partial \pi^*_\nu}, \tag{12}$$

are over-abundant. To eliminate those in excess, let us consider a family of space-like surfaces σ which fills the space-time completely, defined by a parameter τ, and let $n_\nu(x)$ be the versor normal to the generic surface and directed towards the future. Bearing in mind the form of the Hamiltonian (5), equations (12) may be written:

$$n_\nu \frac{\partial\varphi}{\partial x_\nu} = -\frac{1}{4}\pi^*_\nu n_\nu, \quad n_\nu \frac{\partial\varphi^*}{\partial x_\nu} = -\frac{1}{4}\pi_\nu n_\nu. \tag{13}$$

Furthermore:

$$n_\nu \frac{\partial\varphi}{\partial x_\nu} = \frac{\partial(n_\nu\varphi)}{\partial x_\nu}, \qquad n_\nu \frac{\partial\varphi^*}{\partial x_\nu} = \frac{\partial(n_\nu\varphi^*)}{\partial x_\nu}, \tag{14}$$

as is seen immediately, because $n_\nu(x)$ is a « solenoidal field ».

Let us now consider an element of quadrivolume $\Delta\omega = S\,\Delta\tau$, where S is an element of space-like surface and $\Delta\tau$ is a temporal interval. Putting

$$\chi = \pi_\lambda n_\lambda, \qquad \chi^* = \pi^*_\lambda n_\lambda, \tag{15}$$

which we will call momenta conjugate on σ to the Lagrangian variables φ, φ^*; if S is sufficiently small so that χ is constant on S, we may write:

$$\frac{\partial\pi_\lambda}{\partial x_\lambda} = \frac{\delta(\chi s)}{\delta\sigma(x)} = \frac{\mathrm{d}\chi(x)}{\mathrm{d}\tau}, \quad \text{etc.} \qquad \frac{\partial(n_\nu\varphi)}{\partial x_\nu} = \frac{\mathrm{d}\varphi}{\mathrm{d}\tau}, \quad \text{etc.} \tag{16}$$

Therefore, introducing the invariant function

$$H^{(\tau)}_\sigma = -\tfrac{1}{4}\chi^*\chi + 4^2_\mu\varphi^*\varphi, \tag{17}$$

which because of its structure we will call density of energy on σ, the classical Hamiltonian system (I.9) may be written in the covariant form:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \frac{\partial H^{(\pi)}_\sigma}{\partial\chi}, \qquad \frac{\mathrm{d}\chi}{\mathrm{d}\tau} = -\frac{\partial H^{(\pi)}_\sigma}{\partial\varphi}, \quad \text{etc.} \tag{18}$$

Thus the parameterization of the equations of motion is achieved and the over-abundant co-ordinates are eliminated.

As is readily verified, in the $T_{\mu\mu}$ scheme we obtain this result whenever the Hamiltonian is of the form

$$T_{\mu\mu} = aP_{\mu\mu}P_{\mu\mu} + bA_{\mu}A_{\mu}\,, \tag{19}$$

as also in the case of interacting fields whenever the Hamiltonian of interaction does not contain the conjugate momenta.

The physical content of equations (18) is as follows. Having introduced a family of space-like surfaces which completely fill the space-time and the family of curves τ perpendicular to these surfaces (that is, a system of curvilinear co-ordinates) as well as the energy density H_σ on σ, we regard the total energy of the field on σ_0 as being subdivided among an arbitrary number of small cells, each of which surrounds an axis τ. During the motion each cell moves along its axis τ, while the energy contained in it does not vary. In fact, from (18) follows immediately

$$\frac{\mathrm{d}H_\sigma^{(\pi)}}{\mathrm{d}\tau} = 0\,. \tag{20}$$

The less the field spreads out as it moves along τ away from the space-like surface S, the more legitimate the subdivision in small cells becomes; our subdivision is to be understood as an approximation which freezes the degrees of freedom along the space-like surfaces σ, but leaves free those along the lines τ. If the Hamiltonian of interaction between boson fields contains the conjugate momenta, further considerations are necessary. Let us consider the case of a meson field in the presence of a given electromagnetic field. If, in analogy to the procedure for $H^{(\pi)}$, we construct from (8) the density of energy on σ of the meson field in presence of the electromagnetic field, we find:

$$H_\sigma^{(\pi,\gamma\pi)} = -\tfrac{1}{4}P_\alpha^* n_\alpha P_\beta n_\beta + 4\mu^2\varphi^*\varphi + ieA_\alpha n_\alpha P_\beta n_\beta\varphi - ieA_\alpha n_\alpha P_\beta^* n_\beta\varphi^*\,, \tag{21}$$

$$= -\tfrac{1}{4}\chi^*\chi + 4\mu^2\varphi^*\varphi - 4e^2(A_\nu n_\nu)^2\varphi^*\varphi\,, \tag{22}$$

that is, the sum of the energy density $H_\sigma^{(\pi)}$ of the meson field and the Kanesawa-Tomonaga term. Evidently, from (21) we cannot obtain a canonical system of the form (18), but rather the non-canonical system

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \frac{\partial H_\sigma^{(\pi,\gamma\pi)}}{\partial P}\,, \qquad \frac{\mathrm{d}P}{\mathrm{d}\tau} = -\frac{\partial H_\sigma^{(\pi,\gamma\pi)}}{\partial\varphi}\,, \quad \text{etc.,} \tag{23}$$

where $P = P_\nu n_\nu$; that is, the definition of the conjugate momenta on σ is

correct, but the equations of motion are not even approximately correct in the sense mentioned previously.

This shows that the entire Hamiltonian of KANEZAWA and TOMONAGA, containing (2) as well as the part (II), may not be used to treat the classical problem of meson fields and electromagnetic fields in interaction. In fact, the term (2) is the only part of the interaction in the Hamiltonian (22) with which, as we have seen, it is impossible to construct a canonical scheme.

If the Hamiltonian of K.T. is interpreted as an operator to determine the Tomonaga-Schwinger equation, at the limit $\hbar \to 0$, by the W.K.B. method (v. II) we would obtain, in the interaction representation, the afore-mentioned classical problem which is not correct; whereas if the Hamiltonian (8) is taken as this operator these difficulties do not arise at the classical limit.

In the covariant functional formalism developed in II, to pass from the Hamiltonian function (8) to the operator defining the equation of T.S. (I), we put

$$(24) \qquad \mathcal{P}^*_\nu(x) = -\hbar n_\nu \frac{\delta}{\delta\varphi^*(x)}\,, \quad \mathcal{P}_\nu(x) = -\hbar n_\nu \frac{\delta}{\delta\varphi(x)}\,, \quad \mathcal{P}_{\mu\nu}(x) = -\hbar n_\nu \frac{\delta}{\delta A_\mu(x)}\,,$$

while the state vector assumes the functional form

$$(25) \qquad \bar{\Phi}(\varphi^*, \varphi, A_\mu, \sigma) = \bar{B}(\varphi^*, \varphi, A_\mu, \sigma) \exp\left[\frac{i}{\hbar}\bar{V}(\varphi^*, \varphi, A_\mu, \sigma)\right].$$

The equation of motion in the Schrödinger representation may then be written in the usual form (*) (v. (II.21))

$$(26) \qquad \mathcal{H}(x)\bar{\Phi} = i\hbar \frac{\delta\bar{\Phi}}{\delta\sigma(x)}$$

and it is easily seen that the necessary condition for integrability is satisfied:

$$(27) \qquad [\mathcal{H}(x), \mathcal{H}(x')]_- = 0 \qquad x, x' \in \sigma.$$

Furthermore, applying the W.K.B. method developed in II, the real part of (26) becomes, for $\hbar \to 0$:

$$(28) \qquad -\frac{1}{4} n_\lambda^2 \left\{ -\frac{\delta\bar{V}}{\delta\varphi}\frac{\delta\bar{V}}{\delta\varphi^*} - \frac{1}{2}\left(\frac{\delta\bar{V}}{\delta A_\mu}\right)^2 \right\} + 4\mu^2\varphi^*\varphi + \\ + eA_\mu n_\mu \left(\varphi^* \frac{\delta\bar{V}}{\delta\varphi^*} - \varphi\frac{\delta\bar{V}}{\delta\varphi}\right) = -\frac{\delta\bar{V}}{\delta\sigma(x)}\,.$$

(*) The Lorentz condition will be studied in the next paragraph.

If the functional $\overline{V} = \int_\sigma V_\nu(\varphi^*, \varphi, A_\mu, x)\, d\sigma_\nu$ satisfies the conditions (*)

$$(29) \qquad \frac{\delta \overline{V}}{\delta \varphi} = -i \frac{\partial V}{\partial \varphi}\,; \quad \frac{\delta \overline{V}}{\delta \varphi^*} = -i \frac{\partial V}{\partial \varphi^*}\,; \quad \frac{\delta \overline{V}}{\delta A_\mu} = -i \frac{\partial V}{\partial A_\mu}\,,$$

with $iV_\nu(\varphi^*, \varphi, A_\mu, x) = n_\nu V(\varphi^*, \varphi, A_\mu)$ where the function $V(\varphi^*, \varphi, A_\mu)$ is an invariant, then with the precautions contained in I and II about the structure of the conjugate momenta, equation (28) becomes the Jacobi equation of the classical problem.

The imaginary part of (26) may be written:

$$(30) \qquad \frac{1}{4}\left\{\frac{\delta}{\delta A_\mu}\left[\bar{B}^2 \frac{\delta \overline{V}}{\delta A_\mu}\right] + \frac{\delta}{\delta \varphi}\left[\bar{B}^2\left(\frac{\delta \overline{V}}{\delta \varphi^*} + 4eA_\nu n_\nu \varphi\right)\right] + \right.$$
$$\left. + \frac{\delta}{\delta \varphi^*}\left[\bar{B}^2\left(\frac{\delta \overline{V}}{\delta \varphi} - 4eA_\nu n_\nu \varphi^*\right)\right]\right\} = \frac{\delta \bar{B}^2}{\delta \sigma(x)}\,.$$

If we interpret the quantities

$$\bar{B}^2 \frac{\delta \overline{V}}{\delta A_\mu}\,; \quad \bar{B}^2\left(\frac{\delta \overline{V}}{\delta \varphi^*} + 4eA_\nu n_\nu \varphi\right)\,; \quad \bar{B}^2\left(\frac{\delta \overline{V}}{\delta \varphi} - 4eA_\nu n_\nu \varphi^*\right),$$

as the components of the current density of the field $\bar{\Phi}(\varphi^*, \varphi, A_\mu, \sigma)$ then (v. II.29) equation (30) expresses the conservation of probability in the configuration space φ^*, φ, A_μ as the space-like surface σ changes.

We may therefore conclude that: if equations (29) and (30) and the sufficient conditions for the existence and uniqueness of the solution of the Jacobi equation are satisfied, then at the classical limit ($\hbar \to 0$) the solution of the equation of motion (26) relative to the problem of a meson field coupled to an electromagnetic field exists and is unique (**).

4. – Spinor field, free and interacting with an electromagnetic field (Dirac case).

Let us briefly apply the classical Hamiltonian formalism developed in I to the free Dirac field. We take as Lagrangian density the function [10]:

$$(31) \qquad L^{(e)} = -\frac{1}{2}\left\{\bar{\psi}(x)\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + k_0\right)\psi(x) + \psi(x)\left(\gamma_\mu^T \frac{\partial}{\partial x_\mu} - k_0\right)\bar{\psi}(x)\right\},$$

(*) The left side of (II. 27) is to be multiplied by i.
(**) The gauge invariance of (26) will be discussed in a separate note.
[10] J. SCHWINGER: *Phys. Rev.*, **74**, 1439 (1948).

using spinor indices and the symbols and conventions in the cited work by SCHWINGER. The following developments are, as far as possible, analogous to those for boson fields. Whenever we do not explicitly give spinor indices, we will use symbols analogous to those for tensors. The evident covariance of the scheme is thus understood referred to the well-known transformation properties of spinors under the Lorentz transformation.

In base of the definition (I.3), the momenta conjugate to the Lagrangian variables $\psi(x)$, $\overline{\psi}(x)$ are respectively

$$4\frac{\partial L^{(e)}}{\partial(\partial\psi/\partial x_\nu)} = p_\nu = -2\overline{\psi}(x)\gamma_\nu\,, \tag{32}$$

$$4\frac{\partial L^{(e)}}{\partial(\partial\overline{\psi}/\partial x_\nu)} = \overline{p}_\nu = -2\psi(x)\gamma_\nu^T\,. \tag{33}$$

The spur of the canonical tensor then becomes

$$H^{(e)} = p_\mu\frac{\partial\psi}{\partial x_\mu} + \overline{p}_\mu\frac{\partial\overline{\psi}}{\partial x_\mu} - 4L = 2k_0(\overline{\psi}\psi - \psi\overline{\psi})\,. \tag{34}$$

Clearly $H^{(e)}$ is a relativistically invariant quantity: nevertheless it is expressed in terms of Lagrangian variables and not their conjugate momenta. But since the latter are linear combinations of the former, as is evident from (32) and (33), they may easily be introduced in $H^{(e)}$ giving:

$$H^{(e)} = \frac{k_0}{8}(p_\nu\overline{p}_\nu - \overline{p}_\nu p_\nu)\,. \tag{35}$$

To extend the $T_{\mu\mu}$ formalism to the fermion field represented by $H^{(e)}$ in the form (34) or (35), several conventions mentioned in the introduction are necessary.

If we consider ψ, $\overline{\psi}$, p_ν, $\overline{p}_\nu$ as functions which commute, the Hamiltonian $H^{(e)}$ is identically zero. This inconvenience is eliminated if we think that these quantities anticommute and write according to the usual convention

$$\overline{\psi}(x)\psi(x) = -\psi(x)\overline{\psi}(x) - iS(x)\,, \tag{36}$$

$$p_\nu(x)\overline{p}_\nu(x) = -\overline{p}_\nu(x)p_\nu(x) - 4iS(x)\,, \tag{37}$$

where $S(x)$ is a function, singular at the point x, for which we understand spinor indices. We therefore write the Hamiltonian function in the forms:

$$H^{(e)} = H^{(e)}(\overline{\psi},\ \psi) = 4k_0\overline{\psi}(x)\,\psi(x) + 2ik_0\,S(x)\,, \tag{38}$$

$$= H^{(e)}(p_\nu, \overline{p}_\nu) = \frac{k_0}{4}p_\nu(x)\overline{p}_\nu(x) + \frac{ik_0}{2}S(x)\,. \tag{39}$$

We observe that the derivative $\partial H^{(e)}/\partial\psi$ for example, calculated according to the usual rules of derivation, is zero if $H^{(e)}$ is of the form (34) and assumes values differing in sign for the form (38) and the form $H^{(e)}(\psi, \bar{\psi})$ obtainable from (34) through (36). Analogous results hold for the derivatives of $H^{(e)}$ in respect of $\bar{\psi}$, p_ν, $\bar{p}_\nu$. To avoid these contradictions, we will make the convention of writing the bilinear expressions in the forms (38) and (39), that is, with $\bar{\psi}$ to the left of ψ and $\bar{p}_\nu$ to the right of p_ν. Furthermore, when taking the derivative in respect of ψ (or $\bar{\psi}$) we suppose the term ψ (or $\bar{\psi}$) written to the right. On the contrary, when deriving in respect of p_ν (or $\bar{p}_\nu$), we suppose them written with p_ν (or $\bar{p}_\nu$) to the left of the other.

We now consider the Hamiltonian function

$$H^{(e)}(\bar{\psi}, \psi, p_\nu, \bar{p}_\nu) = \tfrac{1}{2}H^{(e)}(\bar{\psi}, \psi) + \tfrac{1}{2}H^{(e)}(p_\nu, \bar{p}_\nu)\,. \tag{40}$$

This depends formally on the Lagrangian variables $\bar{\psi}$, ψ and on their conjugate momenta, and therefore we may write the system of Hamilton's equations (I.9):

$$\frac{\partial\psi}{\partial x_\mu} = \frac{\partial H^{(e)}}{\partial p_\mu} = \frac{k_0}{8}\bar{p}_\mu\,, \tag{41}$$

$$\frac{\partial p_\mu}{\partial x_\mu} = -\frac{\partial H^{(e)}}{\partial\psi} = -2k_0\bar{\psi}\,, \tag{42}$$

$$\frac{\partial\bar{\psi}}{\partial x_\mu} = \frac{\partial H^{(e)}}{\partial\bar{p}_\mu} = -\frac{k_0}{8}p_\mu\,, \tag{43}$$

$$\frac{\partial\bar{p}_\mu}{\partial x_\mu} = \frac{\partial H^{(e)}}{\partial\bar{\psi}} = 2k_0\psi\,. \tag{44}$$

From the definitions (32) and (33) of the conjugate momenta, we see that equations (41) and (44) are identified with the Dirac equation, while (42) and (43) give rise to the adjoint Dirac equation.

This indicates that the second group of Hamilton's equations (43) and (44) is over-abundant. This is in accord with the foregoing; the Dirac equations are of the first order and therefore the system of Hamilton's equations must be degenerate. Formally this depends on the fact that the conjugate momenta are linear combinations of the Lagrangian variables and do not contain derivatives of the latter.

In the case of electrons and positrons interacting with an electromagnetic field, the Lagrangian density is:

$$L = -\frac{1}{2}\bar{\psi}(x)\left[\gamma_\mu\left(\frac{\partial}{\partial x_\mu} - ieA_\mu(x)\right) + k_0\right]\psi(x) - \tag{45}$$

$$- \frac{1}{2}\psi(x)\left[\gamma_\mu^T\left(\frac{\partial}{\partial x_\mu} + ieA(x)\right) - k_0\right]\bar{\psi}(x) - \frac{1}{2}\frac{\partial A_\mu}{\partial x_\nu}\frac{\partial A_\mu}{\partial x_\nu}\,.$$

The conjugate momenta are the same as for the free fields, and the spur of the canonic tensor is:

$$H = H^{(0)} + ie(p_\mu(x)\psi(x) - \bar{p}_\mu(x)\bar{\psi}(x))A_\mu(x) - \tfrac{1}{8}P_{\mu\nu}P_{\mu\nu}\,, \tag{46}$$

where $H^{(0)}$ is given by (40). The Hamilton equations obtained from (46) by (I.9) are the usual equations of electromagnetic potential in the presence of current: $J_\mu = (ie/2)(\bar{\psi}\gamma_\mu\psi - \psi\gamma_\mu^T\bar{\psi})$, and the Dirac equations for electrons and positrons in an external electromagnetic field. But in this case as well we obtain overabundant equations which are the generalizations of (43) and (44). Nevertheless in the following we shall overlook this circumstance, and suppose that for spinor fields, free or in the presence of external fields, the effective canonical variables are $\bar{\psi}$, ψ, p_ν, $\bar{p}_\nu$.

With these premises it is possible to follow all the « classical » developments made in I and II (if the order of the canonical variables and the eventual changes of sign at derivation are accounted for by the conventions given previously). Hence we may conclude that a « classical » covariant Hamiltonian formulation is also possible for a Dirac field coupled to an electromagnetic field.

The passage to the second quantization equation of Tomonaga-Schwinger is effected in the usual way: in the Hamiltonian (46) we introduce the substitutions (II.18) for the electromagnetic field, and the substitutions

$$(\mathcal{P}_\nu(x))_\alpha = -n_\nu\hbar\frac{\delta}{\delta\psi_\alpha(x)}\,, \qquad (\bar{\mathcal{P}}_\nu(x))_\alpha = -n_\nu\hbar\frac{\delta}{\delta\bar{\psi}_\alpha(x)}\,, \tag{47}$$

for the conjugate momenta of the Dirac field (α is a spinor index), noting that the relations (II.15) and (II.20) become anticommutation relations for spinor fields. Further, we introduce the state vector in functional form depending on σ and on the Lagrangian variables $\bar{\psi}$, ψ, A_μ, which may be written in the form (25). An equation of the type (26) is thereby completely defined in this case also. However, the problem of electron-photon interaction is not yet fully described; the Lorentz condition must be introduced.

This is written classically, in our notation

$$P_{\mu\mu}(x) = 0\,. \tag{48}$$

As for the quantum-mechanical formulation, we note that because of (II.17), it cannot be that

$$\hbar n_\mu\frac{\delta}{\delta A_\mu(x)} = 0\,.$$

We write therefore, as is known,

$$n_\mu \frac{\delta \bar{\Phi}}{\delta A_\mu(x)} = 0 , \tag{49}$$

with $\bar{\Phi}$ of the type

$$\bar{\Phi}(\bar{\psi}, \psi, A_\mu, \sigma) = \bar{B}(\bar{\psi}, \psi, A_\mu, \sigma) \exp\left[\frac{i}{\hbar} \bar{V}(\bar{\psi}, \psi, A_\mu, \sigma)\right] . \tag{50}$$

From (49) we then obtain:

$$\hbar n_\mu \frac{\delta \bar{B}}{\delta A_\mu(x)} + i n_\mu \frac{\delta \bar{V}}{\delta A_\mu(x)} \bar{B} = 0 . \tag{51}$$

Since $\bar{B}$ and $\bar{V}$ are real, generally non-vanishing functionals, we must have:

$$n_\mu \frac{\delta \bar{B}}{\delta A_\mu(x)} = 0 , \tag{52}$$

$$n_\mu \frac{\delta \bar{V}}{\delta A_\mu(x)} = 0 . \tag{53}$$

We note at once that the last relation represents the classical limit of the Lorentz condition written in the form (51). Bearing in mind (II.27) and (I.26), this becomes identified with (48). That is, the Lorentz condition imposes no restrictions on the phase of the state vector different from those imposed by requiring the real part of the equation of motion to reduce to the classical Jacobi equation at the limit ($\hbar \to 0$).

Considering a surface $t =$ constant, we find from (52) and (53):

$$\frac{\delta \bar{B}}{\delta A_4(x)} = 0 , \qquad \frac{\delta \bar{V}}{\delta A_4(x)} = 0 ,$$

that is, the state vector is independent of the scalar potential: in general, for generic space-like surfaces, the two relations (52) and (53) or (49) permit the elimination of the « time-like » dependence of the state vector on the potentials.

We may therefore conclude that the Lorentz condition is compatible with the equation of motion, and that if a relation analogous to (30) on the conservation of probability and the sufficient conditions for the existence and uniqueness of the solution of the Jacobi equation are satisfied, at the classical limit ($\hbar \to 0$) the solution of the equation of motion (26) relative to the problem of coupled photon- and Dirac fields exists and is unique.

* * *

The author wishes to thank in particular Professors E. R. CAIANIELLO, M. CINI, and B. F. TOUSCHEK for many stimulating discussions.

RIASSUNTO

In questo lavoro si estende il formalismo funzionale covariante, sviluppato in due precedenti articoli, al caso di campi bosonici accoppiati e di campi fermionici liberi e accoppiati con campi bosonici. Si discute se è sufficiente la conoscenza dell'energia per determinare l'evoluzione dei campi accoppiati e si danno alcune proprietà del vettore di stato che risultano immediatamente dalla formulazione funzionale covariante nella rappresentazione di Schrödinger. Si dimostra al limite classico, $\hbar \to 0$, che la soluzione delle equazioni del moto per campi accoppiati esiste ed è unica.

IL NUOVO CIMENTO Vol. XIII, N. 5 1° Settembre 1959

The Relation between Cosmic-ray Intensity and Geomagnetic Activity During Periods of High and Low Solar Activity.

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(ricevuto il 26 Maggio 1959)

Summary. — The phase relationships between variations in cosmic-ray intensity and geomagnetic activity were studied for the period June, 1957 - August, 1958. The results indicate a definite difference in the relationship for periods of high and of low solar activity. It is suggested that this difference may be due to the more frequent occurrence of severe Forbush decreases during periods of maximum solar activity.

1. – Introduction.

Van Heerden and Thambyaphillai (1) made a comparative study of the 27-day variations in cosmic-ray intensity and geomagnetic activity during the period June 1952-July 1953. They found, by the well-known Chree-method (2), a phase difference of about five days between the occurrence of cosmic-ray intensity maxima and minima and the minima and maxima of K_p, the planetary index of geomagnetic disturbance, respectively, and suggested that the cosmic-ray intensity changes are associated with solar disturbances, which, after a delay of several days produce magnetic storms.

Kane (3) studied the phase relationship between 27-day sequences of K_p maxima and both the diurnal variation and the daily values of cosmic-ray intensity for the period 1951-1953. He found that for some periods and for

(1) I. J. Van Heerden and T. Thambyaphillai: *Phil. Mag.*, **46**, 1238 (1955).
(2) C. Chree: *Trans. Roy. Soc. (London)*, A **212**, 75 (1913).
(3) R. P. Kane: *Phys. Rev.*, **98**, 130 (1955).

some sequences of K_p maxima, the K_p minima are preceded by cosmic-ray maxima by about three days. SIMPSON (4) studied the distribution of K_p values around 41 cosmic-ray maxima which occurred in the period May 1951-November 1952 and found on the average a steady increase of K_p from a minimum one day before to a maximum about two days after the cosmic-ray maximum.

The above studies all relate to a period of very low solar activity.

VENKATESAN (5) studied the phase relationship between cosmic-ray intensity variations and geomagnetic activity for the period 1937-1958 and observed that the K_p minimum (maximum) preceded the corresponding cosmic-ray one in some years and followed in other years, the latter being generally the case during the rising part of the solar cycle and possibly during the early fall.

2. – Present results.

The neutron intensities recorded at Uppsala (Sweden), Leeds (England) and Hermanus (South Africa) for the period June 1957-August 1958 have been used for a similar study of the phase relationship between cosmic-ray intensity and geomagnetic activity during this period of maximum solar activity. The data received from The Max-Planck-Institute (Germany), Rio de Janeiro (South America) and Uganda (Central-Africa) do not cover the entire period of investigation but assisted in the selection of some events.

It was found that in quite a number of cases the date on which the neutron intensity reached an absolute maximum differed by several days for different stations. A study was made

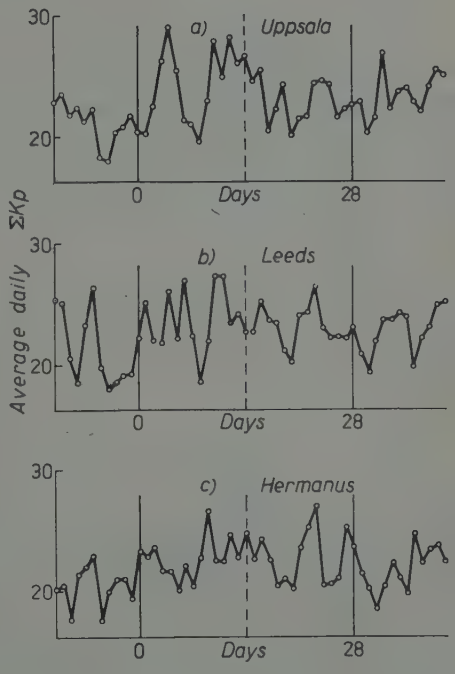

Fig. 1. – Variation of $\sum K_p$ around days of neutron intensity maximum at (*a*) Uppsala, (*b*) Leeds and (*c*) Hermanus selected as zero days.

(4) J. A. SIMPSON: *Phys. Rev.*, **94**, 426 (1954).
(5) D. VENKATESAN: *Suppl. Nuovo Cimento*, **8**, 285 (1958).

of the variation of K_p for a number of days around zero days selected on the basis of absolute maxima in the neutron intensity. This was done separately for each of the stations Uppsala, Leeds and Hermanus. The results are shown in Fig. 1. Fig. 2 shows the result of a similar study but

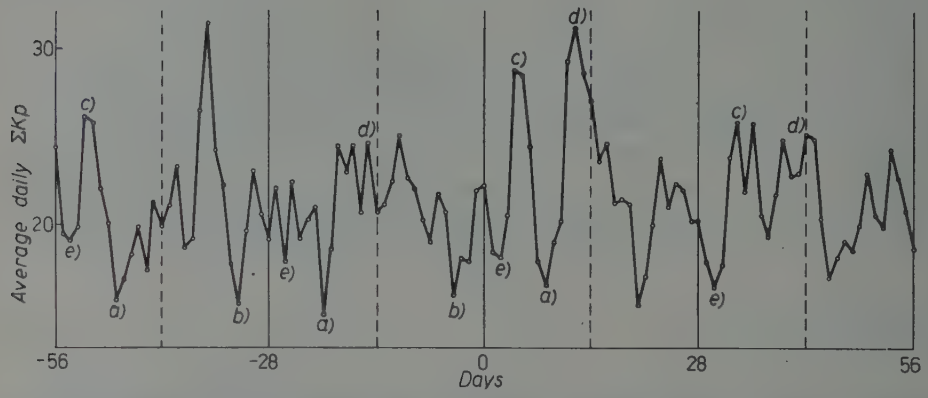

Fig. 2. – Variation of $\sum K_p$ around days of neutron intensity maximum occurring within ± 1 day at least 3 stations selected as zero days.

this time only cases in which the intensity reached a maximum within about three days and at least three stations were considered. The analysis was extended to two periods of 28 days before and after the zero days. The curves of Figs. 1 and 2 all show a certain similarity, although a number of minima and maxima become more prominent in the curve of Fig. 2. The recurrence of quite a number of these minima and maxima is obvious and became clearer after superposition of the four 28-day periods, as can be seen in Fig. 3.

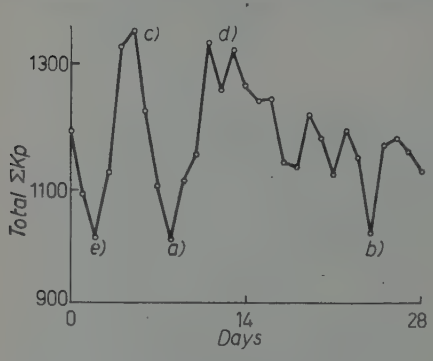

Fig. 3. – Resultant $\sum K_p$-curve for four 28-day periods.

Fig. 4 shows the corresponding Chree-curve for the neutron intensity. The same zero days have been used as for the analysis of Fig. 2. A very clear 27-28 day variation of maximum amplitude $\pm 3\,\%$ is revealed for the neutron intensity during this period of maximum solar activity.

The days on which the neutron intensity reached an absolute minimum were more clearly defined than was the case with maximum days and a comparison showed it to coincide in all cases within ± 1 day for all stations.

The Chree-curves for K_p and neutron intensity, using minimum days of the neutron intensity as zero days, are shown in Fig. 5.

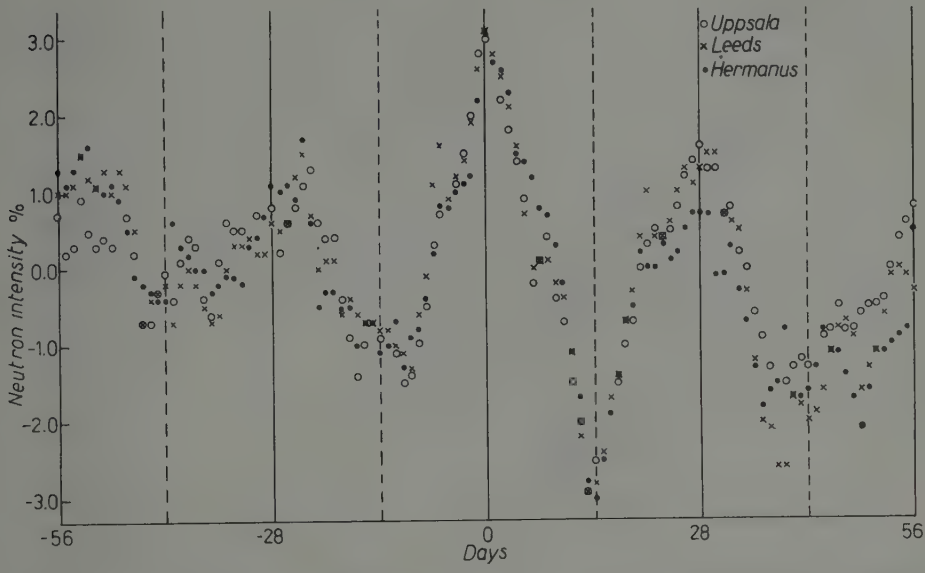

Fig. 4. – Variation of neutron intensity around maximum days selected as zero days.

It thus seems that during this particular period of maximum solar activity the K_p maximum precedes the cosmic-ray minimum by about one day, whilst the behaviour of K_p near cosmic-ray maximum is not at all clear. It is interesting to note however that in Fig. 2 a prominent maximum of K_p appears

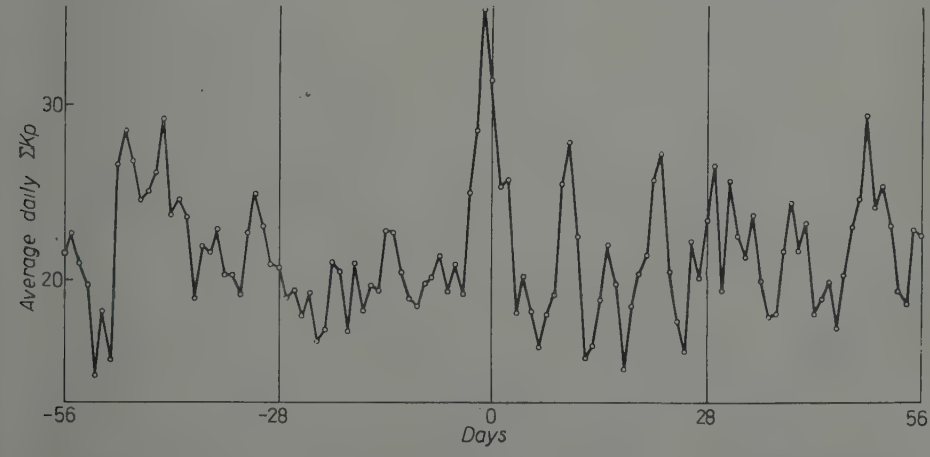

Fig. 5*a*. – Variation of $\sum K_p$ around days of neutron intensity minimum selected as zero days.

about one day before the cosmic-ray minimum occurring around the +14-th day in Fig. 4.

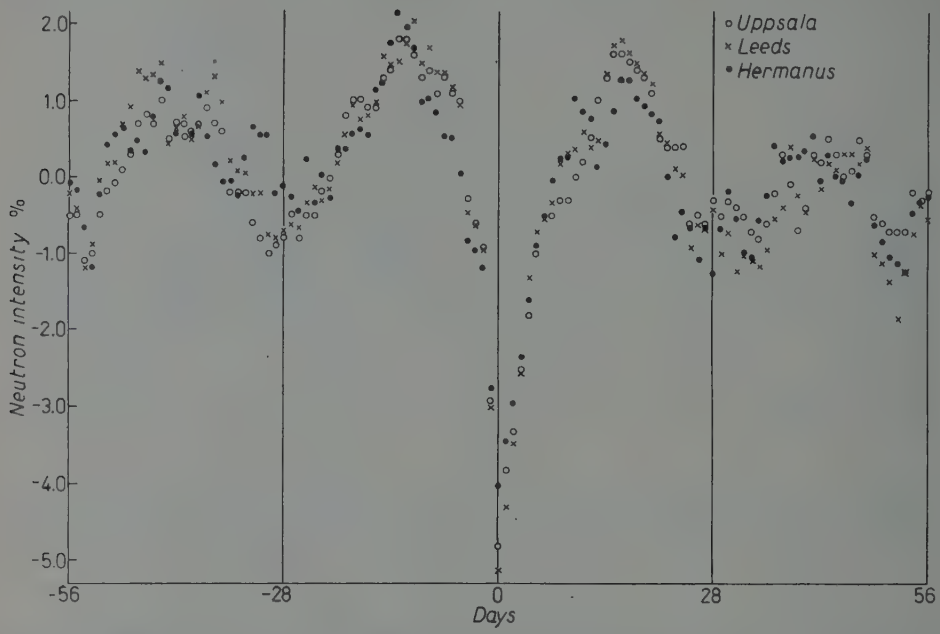

Fig. 5*b*. – Variation of neutron intensity around minimum days selected as zero days.

Fig. 6. shows a comparison of the K_p-curves found by these two procedures of selecting zero days. The upper and lower curves correspond respectively to maximum and minimum days of the neutron intensity as zero days. It is interesting to note the identical peaks appearing in both curves for the two 28-day periods following the zero days.

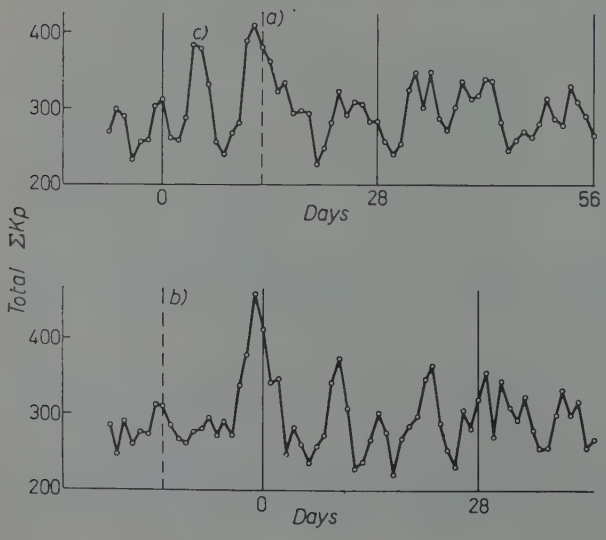

Fig. 6. – Comparison of $\sum K_p$-curves found from maximum and minimum days respectively as zero days. The broken lines (*a*) and (*b*) show the respective positions of a cosmic-ray minimum and of a maximum.

Equally interesting is the absence in the lower curve of the maximum *c*) shown in the upper curve. As can be seen from Figs. 2 and 3, this maximum has a definite recurrence tendency of 27-28 days and follows the cosmic-ray maximum by about 4-5 days. This result is compatible with the findings of KANE and SIMPSON.

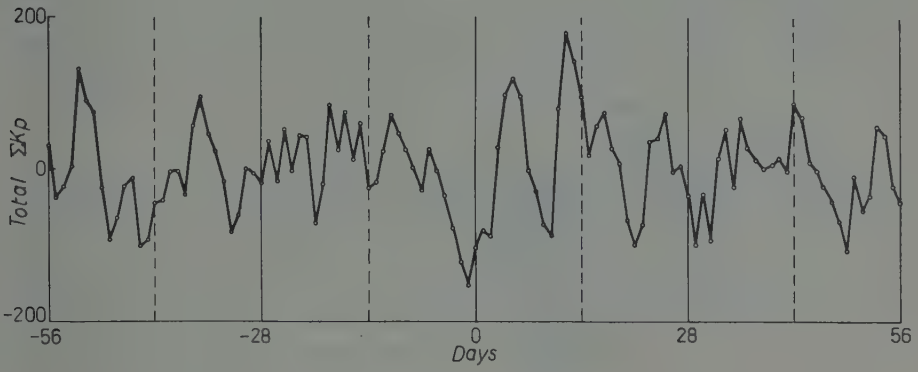

Fig. 7*a*. – Composite $\sum K_p$-curve.

It thus seems as though the curve of Fig. 2 is composed of two definite maxima; the first, *c*), occurring 4-5 days after the cosmic-ray maximum in accordance with previous results, and the second, *d*), appearing near cosmic-ray minimum.

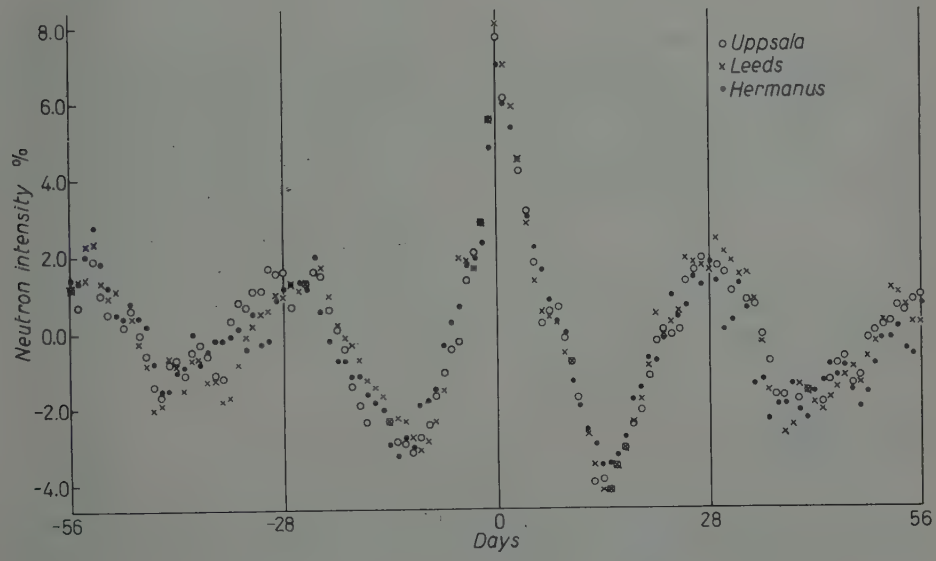

Fig. 7*b*. – Composite neutron intensity curve.

VAN HEERDEN and THAMBYAPHILLAI arrived at their final result by subtracting the curve based on minimum days as zero days from the curve based on maximum days as zero days. Due to the very prominent maximum value of K_p one day before cosmic-ray minimum, an artificial minimum in K_p will be found one day before cosmic-ray maximum if the same be done in the present investigation as seen in Fig. 7. The overall appearance of this curve however does not differ appreciably from the curve in Fig. 2. Comparing the corresponding curve for the neutron intensity in Fig. 7 with the results of VAN HEERDEN and THAMBYAPHILLAI, it is found that the maximum amplitude of the 27-day variation for the present period is about six times that for the period 1952-1953.

SIMPSON [4] selected eight large sudden decreases in the neutron intensity during 1951 and for five of the cases found no evidence of magnetic disturbances near the time when the intensity began to decrease; also that several magnetic « quiet » days occurred during these periods. He thus concluded that the onset of cosmic-ray decreases is not necessarily accompanied by any geomagnetic disturbances.

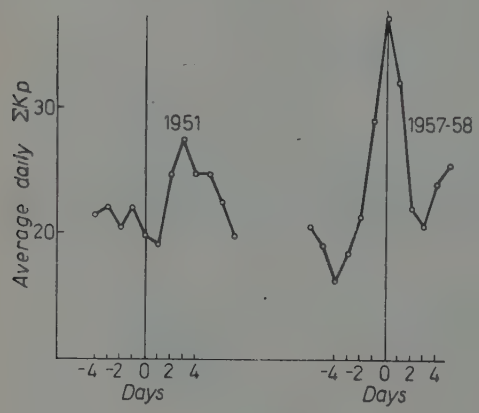

Fig. 8. – Variation of $\sum K_p$ around days of rapid neutron intensity decrease.

These eight decreases in 1951 have now been used to study the distribution of K_p values near them. The result is shown in Fig. 8. We see a maximum of K_p about two days after the onset of these decreases. In Fig. 8 is also shown the K_p variation near the onset of 15 large sudden decreases in the neutron intensity during the period 1957-1958. K_p reaches a very significant maximum value in this case on the day on which the decrease occurs. We thus conclude that whilst for the period of low solar activity (1951) the sudden decreases in cosmic-ray intensity precede geomagnetic disturbances, if any, it seems as though the geomagnetic disturbances and sudden decreases in the cosmic-ray intensity coincide during this period of maximum solar activity.

3. – Conclusion.

We thus see a different relation between cosmic-ray intensity and geomagnetic activity during the last period of low, and the present period of maximum solar activity.

The fact that, for the present period, the cosmic-ray minimum is usually experienced one day after the K_p maximum, suggests that the cosmic-ray decreases associated with magnetie storms (Forbush decreases) are so pronounced that the times of cosmic-ray minima are largely determined by these decreases. The results for the period 1951-1953 on the other hand suggest that for periods of low solar activity the Forbush decreases are, on the average, not of such importance, nor of such a number as to affect the position of the cosmic-ray minima relative to geomagnetic activity.

* * *

The cosmic-ray equipment in operation at the Magnetic Observatory, Hermanus, is the property of the South African Council for Scientific and Industrial Research.

The author is indebted to Prof. Dr. P. H. STOKER of the University of Potchefstroom and Mr. A. M. VAN WIJK, Director of the Hermanus Magnetic Observatory, for suggestions and helpful discussions.

RIASSUNTO (*)

Sono state studiate, per il periodo di tempo intercorrente fra il Giugno 1957 e l'Agosto 1958, le relazioni di fase fra le variazioni dell'intensità dei raggi cosmici e dell'attività geomagnetica. I risultati indicano una differenza definita di tali relazioni per i periodi di intensa e debole attività solare. Si suggerisce che questa differenza possa essere dovuta alla maggiore frequenza di forti decrementi di Forbush durante i periodi di massima attività solare.

(*) *Traduzione a cura della Redazione.*

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Some Remarks on the Calculation of the Polarization Effects - I.

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(ricevuto il 27 Maggio 1959)

Summary. — In connection with the methods developed during the last few years for the study of the effects of polarization in processes concerning photons and $\frac{1}{2}$ spin particles, we consider here a formulation enabling to work out expressions as simple as possible. For the interpretation and discussion of the results it is always advantageous to employ the Stokes parameters; however we have deemed it advisable to use them in their elementary formulation. On the contrary, it is not convenient to have recourse to the calculation of traces of matrices which leads to an unnatural dilatation of the formulae. This observation is particularly useful in processes of a high order. Further, the invariance properties of interactions may be connected to the particular relations existing between the various matrix elements, which will permit further simplifications and controls. As an application, we have carried out the calculations of the effects of polarization in the simple Compton scattering. In a further paper we shall deal with the calculation of the polarization effects in the double Compton scattering.

1. – Introduction.

In the last few years, various formalisms dealing with the effects of polarization in processes concerning photons and $\frac{1}{2}$ spin particles for any degree and type of polarization of the beams interested, have been developed [1-9].

(1) A. Wightman: *Phys. Rev.*, **74**, 1813 (1948).
(2) U. Fano: *Journ. Opt. Am. Soc.*, **39**, 859 (1949).
(3) F. W. Lipps and H. A. Tolhoek: *Physica*, **20**, 85, 395 (1954).
(4) U. Fano: *Phys. Rev.*, **93**, 121 (1954).
(5) M. S. Walker: *Am. Journ. Phys.*, **22**, 170 (1954).
(6) H. A. Tolhoek and R. S. De Groot: *Physica*, **17**, 1 (1951).
(7) W. H. Mc Master: *Am. Journ. Phys.*, **22**, 351 (1954).
(8) H. A. Tolhoek: *Rev. Mod. Phys.*, **28**, 277 (1956).
(9) C. Bouchiat and L. Michel: *Nucl. Phys.*, **5**, 416 (1958).

All these formalisms lead one to express the transition probability in a form which will directly permit the discussion of the various effects of polarization without any further work.

This is obtained, as already done by WIGHTMAN [1] for the Compton effect and later, more generally, by FANO [4], by introducing density matrices in the transition probability. A simple and elegant way to deal with polarization effects—particularly significant for the scattering processes—is to express the transition probability through an operator which, when applied to the polarization state of the incident beam, will give that of the scattered one [2].

In such a case it will be useful to express the states of polarization, instead of by means of the density matrices, through the Stokes parameters [2-8]. Since these four real quantities express the results of the measures to be effected to individuate the polarization states of a beam, they represent it in the most direct and adequate way.

LIPPS and TOLHOEK [3] developed a formalism allowing to express the transition probabilities in terms of the Stokes parameters for all the particles involved in the process and they successfully applied it to the Compton effect obtaining explicitly all the possible correlations between the states of polarization. This formalism is applicable to any process where photons and $\frac{1}{2}$ spin particles are concerned, independently of the conservation of the number of particles. Such a technique was applied by BÖBEL [10] to calculate the effects of polarization in the bremsstrahlung and a similar formalism was developed by KRESNIN and ROSENTSVEIG [11] to calculate the effects of polarization in the Møller scattering. A relativistically invariant formulation to study the effects of polarization in the Møller scattering was developed by BOUCHIAT and MICHEL [9].

The reason for the substantial interest in the results obtained by the above mentioned authors is to express the transition probability as a sum of terms each describing in quite a direct manner the dependence of the cross-section on the state of polarization of one or more particles. To calculate the transition probability in terms of the Stokes parameters, employing the usual perturbation theory, it is necessary, however, to introduce for the electrons a rather complicated 4×4 density matrix; the explicit calculation of the transition probability, which appears as a trace of Dirac and Pauli matrices, is usually rather laborious and practically prohibitive for processes of higher order.

We herein suggest a method which also leads to expressing the transition probability as a sum of terms each describing the dependence of the cross-

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[10] G. BÖBEL: *Nuovo Cimento*, **6**, 1241 (1957).

[11] A. A. KRESNIN and L. N. ROSENTSVEIG: *Journ. Exp. Theor. Phys.*, **5**, 288 (1957) (Russian reference **32**, 353).

section on the state of polarization of one or more particles. Such a method differs from the above mentioned ones because we are not trying to calculate directly the transition probability but we are trying instead to express it ar a sum of particular combinations of the amplitudes for the transitions between the basic states of polarization. Each of these terms describes a particulas effect of polarization. We shall be able to obtain any information relative to any effect of polarization in the process once we know all the amplitudes for the transitions between the basic states for which the calculation is usually rather simple and very often easier than a direct calculation of the transition probabilities depending on the states of polarization of all particles. To obtain the above mentioned expression of the transition probability the use of the Stokes parameters is, here too, particularly useful.

In connection with the Møller scattering, STEHLE ([12]) has already noticed that the simplest way to examine all the polarization effects is to calculate directly the amplitudes for the transitions between the basic states.

We have calculated such amplitudes for the single Compton effect and for the double Compton effect. In the former case the amplitudes will be extremely simple and, on the basis of their knowledge, the calculation of any polarization effect will result almost immediately. The results obtained by LIPPS and TOLHOEK ([3]) are thus easily found again. In the latter case the direct calculation of the transition probability as a function of the states of polarization of all particles is practically prohibitive, whereas the calculation of the amplitudes, or at least their numerical determination, can be obtained fairly easily, and from this we can calculate any polarization effect.

In the two following Sections of this paper, we are giving the rule of combination of the amplitudes and the amplitudes themselves for the single Compton effect. In a subsequent paper we shall derive the amplitudes for the double Compton effect and discuss some polarization effects of particular interest which appear in it.

2. – Deduction of the amplitude combination rule.

Let us now deduce the expression of transition probability which will permit us to obtain immediately the description of any polarization effect once we know the amplitudes for the transitions between the basic states of polarization of all the particles involved. The deduction of such expression does not contain any new conceptual notions related to the polarization effects already developed in the past, and, therefore, we shall omit a description of formalisms already extensively used.

([12]) P. STEHLE: *Phys. Rev.*, **110**, 1458 (1958).

Let us consider a generic electrodynamic process in which m photons and p electrons are in the initial state, n photons and q electrons in the final state.

We can write the amplitude for the transition between the initial and final state in which all particles are polarized in the form:

$$(1) \qquad A_{\mathrm{fin,\,in}} = M\langle \boldsymbol{e}^1, \dots \boldsymbol{e}^n,\ u^1, \dots u^q \,|\, H \,|\, \boldsymbol{e}^{01}, \dots \boldsymbol{e}^{0m},\ u^{01}, \dots u^{0p}\rangle \,.$$

Here H is the interaction Hamiltonian; $M = ((2\pi/\hbar)d_f)^{\frac{1}{2}}$, where d_f is the density of the final states; $\boldsymbol{e}^{0j}, \boldsymbol{e}^i$ are the polarization vectors of the j-th initial photon and of the i-th final photon; u^{0k} and u^h are the spinors representing the polarization of the k-th initial electron and of the h-th final electron.

By introducing the basic states $\boldsymbol{e}_{\alpha_i}$, $\boldsymbol{e}^0_{\beta_j}$, u_{γ_h}, $u^0_{\delta_k}$, $(\alpha, \beta, \gamma, \delta = 1, 2)$ for the photons and the electrons in the initial and final states, we have:

$$(2) \qquad \boldsymbol{e}^i = a_{\alpha_i}\boldsymbol{e}_{\alpha_i}, \qquad \boldsymbol{e}^{0j} = a_{\beta_j}\boldsymbol{e}^0_{\beta_j}, \qquad u^h = b_{\gamma_h}u_{\gamma_h}, \qquad u^{0k} = b^0_{\delta_k}u^0_{\delta_k} \,.$$

(We use the summation convention only for repeated greek indices) and (1) becomes:

$$(3) \qquad \begin{cases} A_{\mathrm{fin,\,in}} = M\langle a_{\alpha_1}\boldsymbol{e}_{\alpha_1}, \dots, a_{\alpha_n}\boldsymbol{e}_{\alpha_n}, b_{\gamma_1}u_{\gamma_1}, \dots, \\ \qquad\qquad \dots, b_{\gamma_q}u_{\gamma_q} \,|\, H \,|\, a^0_{\beta_1}\boldsymbol{e}^0_{\beta_1}, \dots, a^0_{\beta_m}\boldsymbol{e}^0_{\beta_m}, b^0_{\delta_1}u^0_{\delta_1}, \dots, b^0_{\delta_p}u^0_{\delta_p}\rangle \,, \\ \qquad = M\bar{a}_\alpha \bar{b}_\gamma\, a^0_\beta b^0_\delta H^{\alpha\beta\gamma\delta} \,, \end{cases}$$

where the bar denotes the complex conjugate and, for the sake of brevity, we have put:

$$(4) \quad a_\alpha = \prod_{i=1}^{n} a_{\alpha_i}, \text{ etc.}; \quad H^{\alpha\beta\gamma\delta} = \langle \boldsymbol{e}_{\alpha_1}, \dots, \boldsymbol{e}_{\alpha_n}, u_{\gamma_1}, \dots, u_{\gamma_q} \,|\, H \,|\, \boldsymbol{e}^0_{\beta_1}, \dots, \boldsymbol{e}^0_{\beta_m}, u^0_{\delta_1}, \dots, u^0_{\delta_p}\rangle \,.$$

We obtain the transition probability by means of averages and sums of $|A_{\mathrm{fin,\,in}}|^2$ opportunely carried out on the initial and final states according to the polarization effects we wish to consider. Such operations directly introduce into the expression of the transition probability the density matrices. If we write, following FANO [3], such averages and sum operations by $\sum^*_{\mathrm{in}} \sum^*_{\mathrm{fin}}$ we have:

$$w = \sum^*_{\mathrm{in}} \sum^*_{\mathrm{fin}} |A_{\mathrm{fin,\,in}}|^2 = M^2 \varrho_{\alpha'\alpha}\varrho^0_{\beta\beta'}\tau_{\gamma'\gamma}\tau^0_{\delta\delta'} H^{\alpha\beta\gamma\delta}\overline{H}^{\alpha'\beta'\gamma'\delta'} \,,$$

where:

$$\varrho_{\alpha'\alpha} = \prod_{i=1}^{n} \varrho'_{\alpha_i\alpha_i}, \qquad \varrho^0_{\beta\beta'} = \prod_{j=1}^{m} \varrho_{\beta_j\beta'_j}, \qquad \tau_{\gamma'\gamma} = \prod_{h=1}^{q} \tau_{\gamma'_h\gamma_h}, \qquad \tau^0_{\delta\delta'} = \prod_{k=1}^{p} \tau^0_{\delta_k\delta'_k},$$

and $\varrho^0_{\beta_j\beta'_j}$, $\varrho_{\alpha'_i\alpha_i}$ are the density matrices of the j-th initial photon and of the analyzer for the i-th final photon; $\tau^0_{\delta_k\delta'_k}$, $\tau_{\gamma'_h\gamma_h}$ the density matrices of the k-th initial electron and of the analyzer for the h-th final electron.

As the density matrix completely individuates the states of polarization of a beam, we see from (5) that to calculate any polarization effect, we must know the amplitude of transition between the basic states. However it is not immediate, on the basis of (5), to assign the rule by which we must associate the amplitudes of transition to obtain the description of the polarization effect under consideration. This is easily obtained by introducing the Stokes parameters ([2-7]).

The substitution of the density matrices for the Stokes parameters in (5) is immediate if we use the well known relation:

$$\varrho_{ij} = \tfrac{1}{2} P_\lambda (\sigma_\lambda)_{ij} , \tag{6}$$

where ϱ is the density matrix; P_λ ($\lambda = 0, 1, 2, 3$) are the Stokes parameters; σ_0 the unit matrix; σ_1, σ_2, σ_3 are the Pauli spin matrices in the usual representation, with σ_3 diagonal. From (6) it follows that P_0 represents the beam intensity, whereas the physical meaning of the other three parameters also follows immediately from (6) once we have fixed the basic states. These three parameters constitute the components of the Stokes vector $\boldsymbol{P}$.

Thus (5) becomes, indicating with U^{0j} and U^i the Stokes parameters of the j-th initial photon and of the analyzer for the i-th final photon and with V^{0k} and V^h the Stokes parameters of the k-th initial electron and of the analyzer for the h-th final electron:

$$w = \frac{1}{2^{m+n+p+q}} M^2 \prod_{i=1}^{n} \prod_{j=1}^{m} \prod_{h=1}^{q} \prod_{k=1}^{p} U^i_{\lambda_i} U^{0j}_{\mu_j} V^h_{\nu_h} V^{0k}_{\varrho_k} (\sigma_{\lambda_i})_{\alpha'_i\alpha_i} (\sigma_{\mu_j})_{\beta_j\beta'_j} \cdot \\ \cdot (\sigma_{\nu_h})_{\gamma'_h\gamma_h} (\sigma_{\varrho_k})_{\delta_k\delta'_k} H^{\alpha\beta\gamma\delta} \bar{H}^{\alpha'\beta'\gamma'\delta'} , \tag{7}$$

$$(U^i_0 = U^{0j}_0 = V^h_0 = V^{0k}_0 = 1) .$$

The sums on the final states of polarization of a particle are carried out simply by doubling all the terms containing the 0-th parameter of Stokes and suppressing those containing the remaining three.

In a like manner, apart from the division by 2, we effect the average on the initial states of polarization of a particle. This procedure is based on the fact that complementary states of polarization are represented by opposite Stokes vectors.

It follows that the first term of (7) ($\lambda_i = \mu_j = \nu_h = \varrho_k = 0$ for any i, j, h, k) multiplied by 2^{n+q} represents the transition probability irrespective of the polarizations (*i.e.* averaged over the initial states and summed on the final states). Each of the other terms describes a particular effect of polarization of the process: those terms in which only one index λ_i, μ_j, ν_h, ϱ_k is different

from zero describe effects regarding the state of polarization of only one particle; the other terms describe all the possible correlations between the polarization states of the particles. Any correlation can therefore be calculated by isolating in (7) the corresponding term, *i.e.* taking into consideration that term of (7) where the indices λ_i, μ_j, ν_h, ϱ_k have such values that the product $\prod_{i=1}^{n}\prod_{j=1}^{m}\prod_{h=1}^{q}\prod_{k=1}^{p} U^{i}_{\lambda_i} U^{0j}_{\mu_j} V^{h}_{\nu_h} V^{0k}_{\varrho_k}$ correspond to the correlation under consideration.

We see therefore how (7) supplies us with the law of association of the transition amplitudes between the basic states so as to calculate, through these, any correlation between the polarization states.

Finally, we have to bring to notice how (7), without having to work out any explicit calculation, allows us to affirm or exclude the existence of some correlations simply on the basis of the general properties of the matrix elements $H^{\alpha\beta\gamma\delta}$. However in general, we can either calculate the $2^{(m+n+p+q)}$ quantities $H^{\alpha\beta\gamma\delta}$ and then construct two by two the relative products, or calculate straight away the $2^{2(m+n+p+q)}$ quantities $H^{\alpha\beta\gamma\delta}\overline{H}^{\alpha'\beta'\gamma'\delta'}$.

The calculation of the quantities $H^{\alpha\beta\gamma\delta}\overline{H}^{\alpha'\beta'\gamma'\delta'}$ can usually be brought to the calculation of traces. This fact, while on the one hand it allows simplifications since it permits one to employ the properties of traces of Dirac matrices, yields on the other hand some complications as it involves an artificial dilatation of the formulae through the introduction of projection operators.

We shall discuss this point in the appendix, with reference to the example of the single Compton effect.

The calculation of the quantities $H^{\alpha\beta\gamma\delta}$ can be effected directly in a fairly easy manner, and, at least for processes of an order higher than the second one, it seems more advisable to calculate such quantities. In fact we have to deal with half a number of Dirac matrices and no artificial dilatations in the formulae are to be introduced.

Besides, particularly for complicated processes, it may be nearly impossible to determine the explicit analytical dependence of the quanties $H^{\alpha\beta\gamma\delta}\overline{H}^{\alpha'\beta'\gamma'\delta'}$ on all the variables involved in the process, while it can still be possible to determine the quantities $H^{\alpha\beta\gamma\delta}$.

When these are numerically known, we can obtain by means of (7) the description of any polarization effect for those values of the dynamic variables which one wishes to consider.

3. – Single Compton effect amplitudes.

The differential cross-section for the Compton effect is written as follows:

$$\mathrm{d}\sigma/\mathrm{d}\Omega = w/c = (1/c)\sum_{\mathrm{in}}^{*}\sum_{\mathrm{fin}}^{*} |A_{\mathrm{fin,in}}|^2 , \tag{8}$$

where

$$A_{\text{fin, in}} = M_c \langle \boldsymbol{e}, u | H | \boldsymbol{e}^0, u^0 \rangle , \tag{9}$$

with

$$M_c = M(e^2/2\mu)(2\pi\hbar^2 c^2/k_0^{\frac{1}{2}} k^{\frac{1}{2}}) = (e^2/2\mu)(k/k_0)(Ec/\mu)^{\frac{1}{2}} , \qquad \mu = mc^2 \tag{10}$$

and

$$\langle \boldsymbol{e}, u | H | \boldsymbol{e}^0, u^0 \rangle = u^* \left(\frac{\boldsymbol{\alpha}\cdot\boldsymbol{e}\, K' \boldsymbol{\alpha}\cdot\boldsymbol{e}^0}{k_0} - \frac{\boldsymbol{\alpha}\cdot\boldsymbol{e}^0 K'' \boldsymbol{\alpha}\cdot\boldsymbol{e}}{k} \right) u^0 , \tag{11}$$

$$\begin{cases} K' = \mu(1+\beta) + k_0 + \boldsymbol{\alpha}\cdot\boldsymbol{k}_0 , \\ K'' = \mu(1+\beta) - k - \boldsymbol{\alpha}\cdot\boldsymbol{k} . \end{cases} \tag{12}$$

The meanings of the symbols used are the same as those adopted in [13] excepting some factors in the interaction Hamiltonian H which have been included in M_c.

We have chosen as basic states for the initial photons the circular polarization vectors $\boldsymbol{e}_1^0$ and $\boldsymbol{e}_2^0$ of components $(1, i, 0)$ and $(1, -i, 0)$ on three orthogonal fixed axes x, y, z with the z axis in the direction of incidence; and for the final photons, the circular polarization vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ of components $(1, i, 0)$ and $(1, -i, 0)$ on the three axes specified by the unitary vectors $\boldsymbol{n}_1 = [(\boldsymbol{k}_0 \wedge \boldsymbol{k}) \wedge \boldsymbol{k}] / |(\boldsymbol{k}_0 \wedge \boldsymbol{k}) \wedge \boldsymbol{k}|$ $\boldsymbol{n}_2 = (\boldsymbol{k}_0 \wedge \boldsymbol{k}) / |\boldsymbol{k}_0 \wedge \boldsymbol{k}|$, $\boldsymbol{n}_3 = \boldsymbol{k}/|\boldsymbol{k}|$.

For the initial electrons we have chosen the states with the spin parallel and antiparallel to $\boldsymbol{k}_0$, and for the final electrons also longitudinal polarization states. It follows from (6) that U_1^0 describes the polarization in the xz plane, U_2^0 the polarization in the plane rotated around z by $\pi/4$ anticlockwise, U_3^0 the circular polarization. Similarly U_1 describes the polarization in the scattering plane, U_2 the polarization in the plane rotated around $\boldsymbol{k}$ by $\pi/4$, U_3 the circular polarization. As regards the electrons, V_1^0, V_2^0, V_3^0 describe the polarization along the axes x, y, z, and V_1, V_2, V_3 describe the polarization along the directions defined by the vectors $(\boldsymbol{k}_0 \wedge \boldsymbol{p}) \wedge \boldsymbol{p} / |(\boldsymbol{k}_0 \wedge \boldsymbol{p}) \wedge \boldsymbol{p}|$, $(\boldsymbol{k}_0 \wedge \boldsymbol{p}) / |\boldsymbol{k}_0 \wedge \boldsymbol{p}|$, $\boldsymbol{p}/|\boldsymbol{p}|$, $\boldsymbol{p}$ being the momentum of the final electron.

The amplitudes $\langle \boldsymbol{e}_\alpha, u_\gamma | H | \boldsymbol{e}_\beta^0, u_\delta \rangle$ can be easily calculated. By calling α the angle between $\boldsymbol{k}_0$ and $\boldsymbol{k}$, β the azimuth of k, E the total energy of the final electron, N the quantity $[4pE(p+p_z)(E+\mu)]^{-\frac{1}{2}}$, $B_\pm$ the quantity $E + \mu \pm p$,

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[13] W. HEITLER: *The Quantum Theory of Radiation* (Oxford, 3-th ed.) p. 215.

we have:

$$
\left\{
\begin{aligned}
&H^{1111} = \text{c.c. } H^{2222} = iN(p + p_z)B_+(\cos\alpha - 1)\exp[i3\beta/2]\,,\\
&H^{1121} = -\text{c.c. } H^{2212} = -iNkB_-\sin\alpha\,(\cos\alpha - 1)\exp[i3\beta/2]\,,\\
&H^{2211} = \text{c.c. } H^{1122} = iN(p + p_z)B_-(\cos\alpha - 1)\exp[-i\beta/2]\,,\\
&H^{2221} = -\text{c.c. } H^{1112} = -iNkB_+\sin\alpha\,(\cos\alpha - 1)\exp[-i\beta/2]\,,\\
&H^{2111} = \text{c.c. } H^{1222} = iN(p + p_z)B_-\,(\cos\alpha + 1)\exp[i3\beta/2]\,,\\
&H^{2121} = -\text{c.c. } H^{1212} = -iNkB_+\sin\alpha\,(\cos\alpha + 1)\exp[i3\beta/2]\,,\\
&H^{1211} = \text{c.c. } H^{2122} = iN[(p + p_z)B_-\,(\cos\alpha + 1) + 2kp\sin^2\alpha]\exp[-i\beta/2]\,,\\
&H^{1221} = -\text{c.c. } H^{2112} = -iN[kB_+\sin\alpha\,(\cos\alpha + 1) + \\
&\qquad\qquad + 2p(p + p_z)\sin\alpha]\exp[-i\beta/2]\,.
\end{aligned}
\right. \tag{13}
$$

Among these amplitudes there exist the relations:

$$
H^{\alpha\beta\gamma\delta} = \eta\ \text{c. c. } H^{\alpha'\beta'\gamma'\delta'} \qquad \begin{cases} \eta = +1 & \text{for } \gamma = \delta, \\ \eta = -1 & \text{for } \gamma \neq \delta, \end{cases} \tag{14}
$$

with $\alpha \neq \alpha'$, $\beta \neq \beta'$, $\gamma \neq \gamma'$, $\delta \neq \delta'$.

Such relations express the conservation of parity. In fact they allow the equality of the transition probabilities for processes which are obtained, one from the other, by spatial inversion. Also the particular phase relations shown by (14) are essential for the conservation of parity, as it can be easily shown by analysing those processes where, instead of the helicities, states of transverse polarization for the electrons are involved.

The relations between the amplitudes shown by (14) are analogous—as regards the Compton effect—to those obtained by McVoy [14] for the photoelectric effect [15].

The cross-section given by (7) is the sum of many terms each describing a particular polarization effect. It is therefore advisable, using a notation already introduced by Lipps and Tolhoek [3], to set the cross-section in the following form:

$$
(\mathrm{d}\sigma/\mathrm{d}\Omega) = [M_c^2/c][\Phi_0 + \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4]\,, \tag{15}
$$

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[14] K. W. Mc Voy: *Phys. Rev.*, **108**, 365 (1957).

[15] A general theorem on the phase relations between the amplitudes involved by the inversion of the helicities of all particles was established by Lenard (*Phys. Rev.* **107**, 1712 (1957).

where Φ_0 is independent of the polarizations,

$$\Phi_1 = \Phi_1(\boldsymbol{U}^0) + \Phi_1(\boldsymbol{U}) + \Phi_1(\boldsymbol{V}^0) + \Phi_1(\boldsymbol{V}) ,$$
$$\Phi_2 = \Phi_2(\boldsymbol{U}^0, \boldsymbol{U}) + \Phi_2(\boldsymbol{V}^0, \boldsymbol{V}) + \Phi_2(\boldsymbol{U}^0, \boldsymbol{V}^0) + \Phi_2(\boldsymbol{U}^0, \boldsymbol{V}) + \Phi_2(\boldsymbol{U}, \boldsymbol{V}^0) + \Phi_2(\boldsymbol{U}, \boldsymbol{V}) ,$$
$$\Phi_3 = \Phi_3(\boldsymbol{U}^0, \boldsymbol{U}, \boldsymbol{V}^0) + \Phi_3(\boldsymbol{U}^0, \boldsymbol{U}, \boldsymbol{V}) + \Phi_3(\boldsymbol{U}^0, \boldsymbol{V}^0, \boldsymbol{V}) + \Phi_3(\boldsymbol{U}, \boldsymbol{V}^0, \boldsymbol{V}) ,$$
$$\Phi_4 = \Phi_4(\boldsymbol{U}^0, \boldsymbol{V}^0, \boldsymbol{U}, \boldsymbol{V}) .$$

All these Φ can be immediately calculated by means of (7) on the basis of (13) and (14). To make an example, let us develop in detail the calculation of some of the Φ.

Term independent of the polarizations:

$$\Phi_0 = (1/2^4) H^{\alpha\beta\gamma\delta} \overline{H}^{\alpha\beta\gamma\delta} = (1/2E)[\mu(\cos^2\alpha + 1) + (k_0 - k)(1 - \cos\alpha)] .$$

The differential cross-section, averaged on the initial states of polarization and summed on the final states, is $2^2\Phi_0(M_c^2/c)$.

Dependence of the cross-section on the polarization of the initial photon:

$$\Phi_1(\boldsymbol{U}^0) = (1/2^4)[\, U_1^0(H^{\alpha 1\gamma\delta}\overline{H}^{\alpha 2\gamma\delta} + \text{c. c}) - iU_2^0(H^{\alpha 1\gamma\delta}\overline{H}^{\alpha 2\gamma\delta} - \text{c. c}) + U_3^0(H^{\alpha 1\gamma\delta}\overline{H}^{\alpha 1\gamma\delta} - H^{\alpha 2\gamma\delta}\overline{H}^{\alpha 2\gamma\delta})] ,$$
$$= -(1/2)(\mu/E)\sin^2\alpha(U_1^0\cos 2\beta + U_2^0\sin 2\beta) .$$

Dependence of the cross-section on the polarization of the final photon:

$$\Phi_1(\boldsymbol{U}) = (1/2^4)[\, U_1(H^{1\beta\gamma\delta}\overline{H}^{1\beta\gamma\delta} + \text{c. c.}) - iU_2(H^{2\beta\gamma\delta}\overline{H}^{1\beta\gamma\delta} - \text{c. c.}) + U_3(H^{1\beta\gamma\delta}\overline{H}^{1\beta\gamma\delta} - H^{2\beta\gamma\delta}\overline{H}^{2\beta\gamma\delta})] ,$$
$$= -(1/2)(\mu/E)\sin^2\alpha\, U_1 .$$

We thus find the well known dependence of the cross-section on the degree of linear polarization and the polarizing effect of the Compton scattering. The independence on the degree of circular polarization of the incident and diffuse beams readily follows from the relations (14).

Dependence of the cross-section on the polarization of the initial and final electrons: $\Phi_1(\boldsymbol{V}^0) = \Phi_1(\boldsymbol{V}) = 0$ as it can be readily seen from the properties (14). Furthermore the correlation among the linear polarization of the incident photons and the spin of the target electrons is also zero:

$$\Phi_2(U_1^0\boldsymbol{V}^0) = \Phi_2(U_2^0\boldsymbol{V}^0) = 0$$

as it follows from the properties (14). The only existing correlation is easily obtained:

$$\Phi_2(U_3^0 V^0) = (1/2^4)[U_3^0 V_1^0 (H^{\alpha 1\gamma 1}\bar{H}^{\alpha 1\gamma 2} - H^{\alpha 2\gamma 1}\bar{H}^{\alpha 2\gamma 2} + \text{c. c.}) -$$

$$- iU_3^0 V_2^0 (H^{\alpha 1\gamma 1}\bar{H}^{\alpha 1\gamma 2} - H^{\alpha 2\gamma 1}\bar{H}^{\alpha 2\gamma 2} - \text{c. c.}) +$$

$$+ U_3^0 V_3^0 (H^{\alpha 1\gamma 1}\bar{H}^{\alpha 1\gamma 1} - H^{\alpha 1\gamma 2}\bar{H}^{\alpha 1\gamma 2} - H^{\alpha 2\gamma 1}\bar{H}^{\alpha 2\gamma 1} + H^{\alpha 2\gamma 2}\bar{H}^{\alpha 2\gamma 2})] =$$

$$= (U_3^0/2E)(\cos\alpha - 1)[V_1^0 k \sin\alpha\cos\beta + V_2^0 k \sin\alpha \sin\beta + V_3^0(k_0 + k)\cos\alpha] =$$

$$= [(\cos\alpha - 1)/2E] U_3^0 [\boldsymbol{V}^0 \cdot (\boldsymbol{k} + \boldsymbol{k}_0 \cos\alpha)] .$$

This result agrees with that obtained by LIPPS and TOLHOEK [3] and it supplies the principle on which, in the last few years, have been planned the different analysers of circular polarization. In particular the term supplying the correlation between U_3^0 and V_1^0 furnishes the principle on which BELTRAMETTI and VITALE have built their polarimeter [16] and the term supplying the correlation between U_3^0 and V_3^0 furnishes that on which the polarimeter of GUNST and PAGE [17] is based.

In quite a similar way we can rapidly obtain all possible correlations.

* * *

The authors wish to express their gratitude to Prof. A. BORSELLINO for his useful advice.

APPENDIX

To calculate directly the quantities $H^{\alpha\beta\gamma\delta}\bar{H}^{\alpha'\beta'\gamma'\delta'}$ instead of calculating separately the amplitudes, we may proceed in the following manner, which method ultimatimately comes to be the calculation of traces of the Dirac matrices.

We refer to the particular case of the single Compton effect but the procedure can obviously be applied to the other processes. The amplitudes for the transition between the basic states of polarization can be expressed as follows:

$$H^{\alpha\beta\gamma\delta} = u_\gamma^+ Q^{\alpha\beta} u_\delta^0 ,$$

([16]) E. G. BELTRAMETTI and S. VITALE: *Nuovo Cimento*, **9**, 289 (1958).

([17]) S. B. GUNST and L. A. PAGE: *Phys. Rev.*, **92**, 970 (1953).

where $Q^{\alpha\beta}$ is a 4×4 matrix depending on the polarization vectors of the photon, u_γ and u^0_δ are Dirac spinors describing respectively the final and initial states. It follows that:

$$H^{\alpha\beta\gamma\delta}H^{\alpha'\beta'\gamma'\delta'} = u^\dagger_\gamma Q^{\alpha\beta}u^0_\delta u^{0\dagger}_{\delta'}Q^{\dagger\alpha'\beta'}u_{\gamma'} = \sum_{ijhk}(\bar{u}_\gamma)_i Q^{\alpha\beta}_{ij}(u^0_\delta)_j(\bar{u}^0_{\delta'})_h Q^{\dagger\alpha'\beta'}_{hk}(u_{\gamma'})_k\,,$$

$$= \mathrm{Sp}\,\{P^{\gamma'\gamma}Q^{\alpha\beta}P^{0\delta\delta'}Q^{\dagger\alpha'\beta'}\}\,,$$

(we indicate with a bar the complex conjugate and with a cross the hermitian conjugate) where the $P^{0\delta\delta'}$ and the $P^{\gamma'\gamma}$ 4×4 matrices are so defined:

$$(P^{0\delta\delta'})_{jh} = (u^0_\delta)_j(\bar{u}^0_{\delta'})_h\,, \qquad (P^{\gamma'\gamma})_{ki} = (u_{\gamma'})_k(\bar{u}_\gamma)_i\,.$$

To effect the calculation of the traces it will be advisable to express these matrices by means of the projection operators and exchange operators relative to the spin states. If u_1 and u_2 are eigenfunctions of the operator $(\boldsymbol{\sigma}\cdot\boldsymbol{p})/|\boldsymbol{p}|$ we obtain:

$$(1/2)(\boldsymbol{\sigma}\cdot\boldsymbol{n}+1)\begin{cases}u_1\\u_2\end{cases} = \begin{cases}u_1\\0\end{cases},$$

$$(1/2)(\boldsymbol{\sigma}\cdot\boldsymbol{n}-1)\begin{cases}u_1\\u_2\end{cases} = \begin{cases}0\\u_2\end{cases}.$$

Furthermore, for the exchange operators, as it is easy to verify, we have:

$$(1/2)\beta(\boldsymbol{\sigma}\cdot\boldsymbol{l}+i\boldsymbol{\sigma}\cdot\boldsymbol{m})\begin{cases}u_1\\u_2\end{cases} = \begin{cases}0\\u_1\end{cases},$$

$$(1/2)\beta(\boldsymbol{\sigma}\cdot\boldsymbol{l}-i\boldsymbol{\sigma}\cdot\boldsymbol{m})\begin{cases}u_1\\u_2\end{cases} = \begin{cases}u_2\\0\end{cases},$$

where $\boldsymbol{l}$, $\boldsymbol{m}$, $\boldsymbol{n}$ are unitary vectors in the directions $(\boldsymbol{z}\wedge\boldsymbol{p})\wedge\boldsymbol{p}$, $\boldsymbol{z}\wedge\boldsymbol{p}$ and $\boldsymbol{p}$ respectively.

By introducing also the usual projection operator on the positive energy states $\Lambda_+(p) = (H+E)/2|E|$, we obtain:

$$P^{0\delta\delta'} = \Sigma_{\delta\delta'}(p_0)\Lambda_+(p_0)\beta\,, \qquad P^{\gamma'\gamma} = \Sigma_{\gamma'\gamma}(p)\Lambda_+(p)\beta\,,$$

where

$$\Sigma(p) = \frac{1}{2}\begin{pmatrix}\boldsymbol{\sigma}\cdot\boldsymbol{n}+1 & \beta(\boldsymbol{\sigma}\cdot\boldsymbol{l}+i\boldsymbol{\sigma}\cdot\boldsymbol{m})\\ \beta(\boldsymbol{\sigma}\cdot\boldsymbol{l}-i\boldsymbol{\sigma}\cdot\boldsymbol{m}) & \boldsymbol{\sigma}\cdot\boldsymbol{n}-1\end{pmatrix}.$$

It finally follows:

$$H^{\alpha\beta\gamma\delta}\bar{H}^{\alpha'\beta'\gamma'\delta'} = \mathrm{Sp}\,\{\Sigma_{\gamma'\gamma}(p)\Lambda_+(p)\beta Q^{\alpha\beta}\Sigma_{\delta\delta'}(p_0)\Lambda_+(p_0)\beta Q^{\dagger\alpha'\beta'}\}\,.$$

This formula allows us to calculate the quantity $H^{\alpha\beta\gamma\delta}\overline{H}^{\alpha'\beta'\gamma'\delta'}$ by means of the trace of products of Dirac matrices. It also shows that, at least in processes of a high order, the calculation of the quantities $H^{\alpha\beta\gamma\delta}\overline{H}^{\alpha'\beta'\gamma'\delta'}$ may not be convenient owing to the dilatation caused by the introduction of the projection and exchange operators.

RIASSUNTO

In relazione ai metodi sviluppati in questi ultimi anni per lo studio degli effetti di polarizzazione in processi concernenti fotoni e particelle di spin $\frac{1}{2}$, si cerca nel presente lavoro di ottenere una formulazione che consenta di manipolare espressioni le più semplici possibile. Per l'interpretazione e la discussione dei risultati è sempre vantaggioso l'uso dei parametri di Stokes, per cui si è ritenuto sufficiente usarli nella loro formulazione elementare. Non è invece conveniente ridursi al calcolo di tracce di matrici, che porta ad una non naturale dilatazione delle formule. Questa osservazione è particolarmente utile per processi di ordine elevato. Inoltre le proprietà di invarianza delle interazioni si possono collegare a particolari relazioni tra i vari elementi di matrice che intervengono, il che consente ulteriori semplificazioni e controlli. Come applicazione è stato eseguito il calcolo degli effetti di polarizzazione sullo scattering Compton semplice. In un successivo lavoro verranno esposti i risultati del calcolo degli effetti di polarizzazione nello scattering Compton doppio.

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Effective Potential Approach to the Threshold Behaviour of Cross Sections (*).

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Summary. — The study of the energy dependence of scattering and reaction cross sections near thresholds is performed by a method which has its basis in the construction of the effective potentials in every single channel. The formalism employed presents some advantage with respect to other kind of approaches in that it gives a simple physical description of the anomaly (cusp or rounded step) which appears in the scattering cross section at the opening up of a new channel. Furthermore, with this method it is effectively possible to give an explanation why the energy derivative of the scattering cross section is infinite when the threshold is approached from below. The effect is found to be purely quantum mechanical.

1. – Introduction.

Originating from an earlier work by Wigner (1), several papers (2-7) have been published recently on the behaviour of the scattering and reaction cross-sections at the opening up of a new channel.

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(**) On leave of absence from Trieste University on a Fulbright traveling grant.

(1) E. P. Wigner: *Phys. Rev.*, **73**, 1002 (1948).

(2) A. I. Baz: *Journ. Exp. Theor. Phys.*, **6**, 709 (1958).

(3) G. Breit: *Phys. Rev.*, **107**, 1612 (1957).

(4) R. G. Newton: *Ann. Phys.*, **4**, 29 (1958) and *Phys. Rev.* (in press).

(5) R. K. Adair: *Phys. Rev.*, **111**, 632 (1958).

(6) L. Fonda and R. G. Newton: *Ann. Phys.* (in press).

(7) L. Fonda and R. G. Newton: *ΛK production near the Σ threshold, Washington Meeting of the Amer. Phys. Soc.*, April 1959 (to be published).

Common result of these papers is the observation that at the energy for the production of the new channel both scattering and production cross-sections for the already opened channels exhibit an infinite energy derivative. The characteristic anomaly can take the form of a cusp or a rounded step.

As has been emphasized, this kind of effect can give a good method for the experimental determination of the parities and spin of the particles involved in the reactions, and of the scattering phase shifts ([2,4,5,7]).

It is known that, strictly mathematically speaking, no infinite energy derivative should occur when the opened up channel consists of two charged particles. However, as discussed at length in ref. ([6]), the Coulomb effects, which are able to destroy the cusp or rounded step, are either too small for most practical purposes or too high an energy resolution is required in order to get observable effects.

As also pointed out in ref. ([6]), when the newly opened channel is a continuum, consisting therefore of three (or more) particles, then the energy derivative of the integrated production cross-section (the integration extended over the available final energy for one of the particles) does not present any infinity at threshold. Consequently also the other cross-sections should not experience any anomaly in their energy derivatives.

It is the purpose of this paper to give a different approach to this kind of problems, which will have its basis in the construction of the equivalent potentials which are effective in each channel. These potentials will clearly be complex, due to the physical possibility of transitions from one channel to another at the appropriate energy, and non local.

The main advantage of this kind of approach is that, as we shall see, a simple physical description of the whole phenomenon is given and an explanation is effectively found for the infinity in the energy derivative of the scattering cross-section when the threshold is approached from below.

The method employed is reminiscent of the resonance scattering treatment given by DIRAC ([8]). Equations of the type discussed in this paper have been considered recently for other purposes by CINI and FUBINI ([9]) and FESHBACH ([10]).

We will consider in detail the neutral discrete case (two neutral particles in the newly produced channel), but, as stressed above, the qualitative and quantitative feature of the problem will be practically unaltered for the case of two charged particles in the newly opened channel.

For the sake of clarity the discussion will be led for two channels only, the incoming channel and the production channel. The generalization to three

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([8]) P. A. M. DIRAC: *Quantum Mechanics*, 1st ed. (Oxford, 1930), p. 193.
([9]) M. CINI and S. FUBINI: *Nuovo Cimento*, **2**, 75 (1955).
([10]) H. FESHBACH: *Ann. Phys.*, **5**, 357 (1958).

or more channels can be given, but the mathematical problem will be much more involved.

For simplicity, only the case in which the diagonal potential in the production channel is spherically symmetric will be taken into consideration. It is to be emphasized, however, that the method is applicable in general and the same results are expected.

In Section **2** the construction of the potentials effective in each channel is given. The problem is solved exactly and gives rise to a discussion in terms of the continuity equation for the current densities of the two considered channels.

The expressions for the scattering and reaction amplitudes are given in Section **3**. In Sections **4**, **5** and **6** the discussion of their energy derivatives can be found.

2. – Construction of the effective potentials and interpretation.

We will consider the Schrödinger equation for a two-body system which has the possibility of being in only two orthogonal states, in two « channels », the channel 1 described by the wave function ψ_1 and the channel 2 described by the wave function ψ_2.

The equation is:

$$(E - H^{(0)})\psi = V\psi \,, \tag{2.1}$$

where

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad H^{(0)} = \begin{pmatrix} H_1^{(0)} & 0 \\ 0 & H_2^{(0)} \end{pmatrix}, \qquad V = \begin{pmatrix} V_1 & V_{12} \\ V_{21} & V_2 \end{pmatrix}.$$

Since the potential V is hermitian we will have $V_{12} = V_{21}^*$. The elements of V are energy independent. $H_i^{(0)}$ is the expression for the kinetic energy which includes the rest energy for the considered channel.

We can easily separate this equation into two equations whose simultaneous solution gives all the details of the problem.

The equations are given as follows:

$$\begin{cases} (E - H_1^{(0)})\psi_1 = V_1\psi_1 + V_{12}\psi_2 \,, \\ (E - H_2^{(0)})\psi_2 = V_{21}\psi_1 + V_2\psi_2 \,, \end{cases} \tag{2.2}$$

We will take into consideration only the case in which there exists a threshold for the second channel. Our problem is then specified by the following boundary conditions:

1) The channel 1 consists of an incoming wave packet and an outgoing spherical wave.

2) The channel 2 consists of only outgoing waves.

These boundary conditions are mathematically expressed by the formulae (ε is an infinitesimal positive number):

$$\text{(2.3)} \qquad \begin{cases} \psi_1 = \varphi_1^{(0)} + (E + i\varepsilon - H_1^{(0)})^{-1}[V_1\psi_1 + V_{12}\psi_2] \\ \psi_2 = (E + i\varepsilon - H_2^{(0)})^{-1}[V_{21}\psi_1 + V_2\psi_2] \end{cases}$$

or alternatively by:

$$\text{(2.4)} \qquad \begin{cases} \psi_1 = \varphi_1 + (E + i\varepsilon - H_1^{(0)} - V_1)^{-1}V_{12}\psi_2 \\ \psi_2 = (E + i\varepsilon - H_2^{(0)} - V_2)^{-1}V_{21}\psi_1 \end{cases}$$

or combinations of these two.

The wave functions $\varphi_1^{(0)}$ and φ_1 satisfy the equations:

$$\text{(2.5)} \qquad \begin{cases} (E - H_1^{(0)})\varphi_1^{(0)} = 0\,, \\ (E - H_1^{(0)} - V_1)\varphi_1 = 0\,. \end{cases}$$

It is clear that the equations (2.4) are more convenient when one desires to eliminate from one of the equations (2.2) the term which gives the connection to the other equation. Substituting in fact (2.4*b*) in equation (2.2*a*) and (2.4*a*) in equation (2.2*b*) we get:

$$\text{(2.6)} \qquad \begin{cases} (E - H_1^{(0)})\psi_1 = V_1\psi_1 + V_{12}(E + i\varepsilon - H_2^{(0)} - V_2)^{-1}V_{21}\psi_1\,, \\ (E - H_2^{(0)})\psi_2 = V_2\psi_2 + V_{21}(E + i\varepsilon - H_1^{(0)} - V_1)^{-1}V_{12}\psi_2 + V_{21}\varphi_1\,. \end{cases}$$

In order to give the interpretation of the various terms appearing on the right hand side of equations (2.6), it is useful to project these equations on the co-ordinate space. Observing that

$$(E + i\varepsilon - H^{(0)} - V)^{-1} = P(E - H^{(0)} - V)^{-1} - i\pi\,\delta(E - H^{(0)} - V)\,,$$

where P indicates, as usual, that the principal value has to be taken in the integration, we get:

$$\text{(2.7)} \qquad (E - H_1^{(0)}(-i\hbar\nabla))\,\psi_1(\boldsymbol{r}) = \int \mathrm{d}^3r'[V_R^{(1)}(\boldsymbol{r}\,|\,\boldsymbol{r}') - iV_I^{(1)}(\boldsymbol{r}\,|\,\boldsymbol{r}')]\,\psi_1(\boldsymbol{r}')\,,$$

$$\text{(2.8)} \qquad (E - H_2^{(0)}(-i\hbar\nabla))\,\psi_2(\boldsymbol{r}) = \int \mathrm{d}^3r'[V_R^{(2)}(\boldsymbol{r}\,|\,\boldsymbol{r}') - iV_I^{(2)}(\boldsymbol{r}\,|\,\boldsymbol{r}')]\,\psi_2(\boldsymbol{r}') + V_{21}(\boldsymbol{r})\,\varphi_1(\boldsymbol{r})\,,$$

where the potentials V_R and V_I are given by:

$$(2.9)\qquad \begin{cases} V_R^{(1)}(\boldsymbol{r}\,|\,\boldsymbol{r}') = V_1(\boldsymbol{r})\,\delta^3(\boldsymbol{r}-\boldsymbol{r}') + V_{12}(\boldsymbol{r})\langle \boldsymbol{r}\,|\,P(E-H_2^{(0)}-V_2)^{-1}\,|\,\boldsymbol{r}'\rangle V_{21}(\boldsymbol{r}')\,, \\ V_I^{(1)}(\boldsymbol{r}\,|\,\boldsymbol{r}') = \pi V_{12}(\boldsymbol{r})\langle \boldsymbol{r}\,|\,\delta(E-H_2^{(0)}-V_2)\,|\,\boldsymbol{r}'\rangle V_{21}(\boldsymbol{r}')\,, \\ V_{R,I}^{(2)} \qquad = V_{R,I}^{(1)}(1\leftrightarrow 2)\,. \end{cases}$$

Clearly $V_R^{(1.2)}$ and $V_I^{(1,2)}$ are in general non local and energy dependent. $V_I^{(1,2)}$ represents the real absorption which takes place in one channel and which brings « probability » into the other. $V_I^{(1)}$, in our hypothesis of the existence of a threshold for the channel 2, will be zero for energies E below the threshold since $\delta(E-H_2^{(0)}-V_2)$ does not give any contribution n this case.

The second term of $V_R^{(1.2)}$ is just the result of the virtual transitions from one channel to the other and back to the former.

In order to discuss the inhomogeneous term in equation (2.8), and to exploit other properties of equations (2.7) and (2.8), it is convenient to take into consideration the continuity equations for the current density. We have:

$$(2.10)\qquad \frac{\partial \varrho_1}{\partial t} + \operatorname{div} \boldsymbol{j}_1 = -\frac{i}{\hbar}\int \mathrm{d}^3r'[\psi_1^*(\boldsymbol{r})\,V_R^{(1)}(\boldsymbol{r}\,|\,\boldsymbol{r}')\psi_1(\boldsymbol{r}') - \text{c.c.}] - \\ -\frac{1}{\hbar}\int \mathrm{d}^3r'[\psi_1^*(\boldsymbol{r})\,V_L^{(1)}(\boldsymbol{r}\,|\,\boldsymbol{r}')\psi_1(\boldsymbol{r}') + \text{c.c.}]\,,$$

$$(2.11)\qquad \frac{\partial \varrho_2}{\partial t} + \operatorname{div} \boldsymbol{j}_2 = -\frac{i}{\hbar}\int \mathrm{d}^3r'[\psi_2^*(\boldsymbol{r})\,V_R^{(2)}(\boldsymbol{r}\,|\,\boldsymbol{r}')\psi_2(\boldsymbol{r}') - \text{c.c.}] - \\ -\frac{1}{\hbar}\int \mathrm{d}^3r'[\psi_2^*(\boldsymbol{r})\,V_I^{(2)}(\boldsymbol{r}\,|\,\boldsymbol{r}')\psi_2(\boldsymbol{r}') + \text{c.c.}] - \frac{i}{\hbar}[\psi_2^*(\boldsymbol{r})\,V_{21}(\boldsymbol{r})\varphi_1(\boldsymbol{r}) - \text{c.c.}]\,.$$

Analysing the term in $V_R^{(1,2)}$ on the right hand side of equations (2.10) and (2.11), we realize that when integrated over the whole r-space, it vanishes. Consequently, this term does not represent real absorption or real emission, although it gives rise to virtual creations and annihilations in the non local region of interaction.

We point out that the term in $V_I^{(1,2)}$ corresponds to the imaginary part of an optical potential; in fact it is roughly proportional to the density of the particles. However, as discussed by CINI and FUBINI [9] and, more extensively, by FESHBACH [10], we cannot use in a complex potential model a so wildly energy dependent potential like $V_R - iV_I$ given here. Instead, a complex potential model can be defined after having taken a suitable average of the transition matrix.

We see that the inhomogeneous term gives rise to a source term which, broadly speaking, has the form of a matrix element from the initial wave packet φ_1 to the final outgoing wave ψ_2. We observe that this term is different from zero for every energy of the incoming beam, also for energies below the threshold for channel 2. Considering in fact for this case, for simplicity, its integral over the whole r-space, making use of expression (2.4), we get:

$$(2.12) \qquad -\frac{i}{\hbar}\int \mathrm{d}^3 r\, \psi_2^*(\boldsymbol{r})\, V_{21}(\boldsymbol{r})\varphi_1(\boldsymbol{r}) + \text{c.c.} = -\frac{i}{\hbar}\langle \psi_2 | V_{21} | \varphi_1 \rangle + \text{c.c.} =$$

$$= \frac{2\pi}{\hbar}\langle \psi_1 | V_{12} \frac{P}{E - H_2^{(0)} - V_2} V_{21}\, \delta(E - H_1^{(0)} - V_1) V_{12} \frac{P}{E - H_2^{(0)} - V_2} V_{21} | \psi_1 \rangle ,$$

which is clearly different from zero.

Finally, as it should be expected, the integration over the whole r-space of the right hand side of equations (2.10) and (2.11) respectively, vanishes for energies E less than threshold. Thus, the physical condition of overall conservation of particles in channel 1 is satisfied, when the energy is less than the minimum for the real production of particles in channel 2. We note that in particular the integration over the whole r-space of the term containing $V_I^{r(2)}$ in equation (2.11) is, for this case, exactly the opposite of the integrated term in φ_1, *i.e.* of the expression (2.12).

3. – Expression for the scattering and reaction amplitudes.

Let us write the asymptotic expressions for the wave functions ψ_1 and ψ_2 in co-ordinate space:

$$(3.1) \qquad \psi_{1\infty}(\boldsymbol{r}) = \exp[i\boldsymbol{k}_1 \cdot \boldsymbol{r}] + \Theta_1(\boldsymbol{k}_1, \boldsymbol{k}_1') \frac{\exp[ik_1 r]}{r} ,$$

$$(3.2) \qquad \psi_{2\infty}(\boldsymbol{r}) = \qquad\qquad \Theta_2(\boldsymbol{k}_1, \boldsymbol{k}_2) \frac{\exp[ik_2 r]}{r} ,$$

where: $\hbar\boldsymbol{k}_1$ specifies the center of mass momentum of the incoming beam,
$\hbar\boldsymbol{k}_1'$ the momentum of the elastically scattered particles ($|\boldsymbol{k}_1'| = |\boldsymbol{k}_1|$),
$\hbar\boldsymbol{k}_2$ the momentum of the produced channel.

The scattering and reaction cross-sections are related to the scattering and reaction amplitudes Θ_1 and Θ_2 by the definitions:

$$(3.3) \qquad \frac{\mathrm{d}\sigma_1}{\mathrm{d}\Omega_1}(\boldsymbol{k}_1, \boldsymbol{k}_1') = |\Theta_1(\boldsymbol{k}_1, \boldsymbol{k}_1')|^2 ,$$

$$(3.4) \qquad \frac{\mathrm{d}\sigma_2}{\mathrm{d}\Omega_2}(\boldsymbol{k}_1, \boldsymbol{k}_2) = |\Theta_2(\boldsymbol{k}_1, \boldsymbol{k}_2)|^2 \frac{v_2}{v_1} ,$$

where v_i is the relative velocity in the center of mass system for the specified channel.

In order to discuss the energy dependence of the scattering and reaction cross-sections at the threshold for the channel 2, we have to express Θ_1 and Θ_2 in terms of the effective potentials derived in Section 2. After taking the limit $r \to \infty$ of the integral equations corresponding to the equations (2.7) and (2.8), we get the amplitudes Θ_1 and Θ_2 in the form:

$$\Theta_1(\boldsymbol{k}_1, \boldsymbol{k}_1') = -\frac{\mu_1}{2\pi\hbar^2}\int \mathrm{d}^3r\,\mathrm{d}^3r' \exp[-i\boldsymbol{k}_1'\cdot\boldsymbol{r}]\cdot \cdot[V_R^{(1)}(\boldsymbol{r}\,|\,\boldsymbol{r}') - iV_I^{(1)}(\boldsymbol{r}\,|\,\boldsymbol{r}')]\psi_1(\boldsymbol{r}')\,, \tag{3.5}$$

$$\Theta_2(\boldsymbol{k}_1, \boldsymbol{k}_2) = -\frac{\mu_2}{2\pi\hbar^2}\int \mathrm{d}^3r\,\mathrm{d}^3r' \exp[-i\boldsymbol{k}_2\cdot\boldsymbol{r}]\cdot \cdot([V_R^{(2)}(\boldsymbol{r}\,|\,\boldsymbol{r}') - iV_I^{(2)}(\boldsymbol{r}\,|\,\boldsymbol{r}')]\psi_2(\boldsymbol{r}') + V_{21}(\boldsymbol{r})\varphi_1(\boldsymbol{r})\delta^3(\boldsymbol{r}-\boldsymbol{r}'))\,, \tag{3.6}$$

with $V_R^{(1,2)}$ and $V_I^{(1,2)}$ given by (2.9). μ_1 and μ_2 are the reduced masses in channels 1 and 2, respectively.

We introduce the two complete sets of eigenfunctions of the operators $H_1^{(0)}+V_1$ and $H_2^{(0)}+V_2$, satisfying the uncoupled eigenvalue problems:

$$\begin{cases} (H_1^{(0)} + V_1)\varphi_1(E', \alpha') = E'\varphi_1(E', \alpha')\,, \\ (H_2^{(0)} + V_2)\varphi_2(E'', \beta'') = E''\varphi_2(E'', \beta'')\,, \end{cases} \tag{3.7}$$

α and β stand for the observables that together with the energy form a complete set of commuting observables in the first and second case, respectively.

Making use of the completeness relations, we get:

$$\Theta_1(\boldsymbol{k}_1, \boldsymbol{k}_1') = -\frac{\mu_1}{2\pi\hbar^2}\int \mathrm{d}^3r\,\mathrm{d}^3r' \exp[-i\boldsymbol{k}_1'\cdot\boldsymbol{r}]\cdot\Big\{V_1(\boldsymbol{r})\psi_1(\boldsymbol{r})\delta^3(\boldsymbol{r}-\boldsymbol{r}') + \tag{3.8}$$

$$+ V_{12}(\boldsymbol{r})\int \mathrm{d}E''\,\mathrm{d}\beta''\varphi_2(\boldsymbol{r}, E'', \beta'')\varphi_2^*(\boldsymbol{r}', E'', \beta'')\frac{P}{E-E''}V_{21}(\boldsymbol{r}')\psi_1(\boldsymbol{r}') -$$

$$- i\pi\theta(E)V_{12}(\boldsymbol{r})\int \mathrm{d}\beta''\varphi_2(\boldsymbol{r}, E, \beta'')\varphi_2^*(\boldsymbol{r}', E, \beta'')V_{21}(\boldsymbol{r}')\psi_1(\boldsymbol{r}')\Big\}\,,$$

$$\Theta_2(\boldsymbol{k}_1, \boldsymbol{k}_2) = -\frac{\mu_2}{2\pi\hbar^2}\int \mathrm{d}^3r\,\mathrm{d}^3r' \exp[-i\boldsymbol{k}_2\cdot\boldsymbol{r}]\cdot\Big\{V_2(\boldsymbol{r})\psi_2(\boldsymbol{r})\delta^3(\boldsymbol{r}-\boldsymbol{r}') + \tag{3.9}$$

$$+ V_{21}(\boldsymbol{r})\int \mathrm{d}E'\,\mathrm{d}\alpha'\varphi_1(\boldsymbol{r}, E', \alpha')\varphi_1^*(\boldsymbol{r}', E', \alpha')\frac{P}{E-E'}V_{12}(\boldsymbol{r}')\psi_2(\boldsymbol{r}') -$$

$$- i\pi V_{21}(\boldsymbol{r})\int \mathrm{d}\alpha'\varphi_1(\boldsymbol{r}, E, \alpha')\varphi_1^*(\boldsymbol{r}', E, \alpha')V_{12}(\boldsymbol{r}')\psi_2(\boldsymbol{r}') + V_{21}(\boldsymbol{r})\varphi_1(\boldsymbol{r}, E, \alpha)\delta^3(\boldsymbol{r}-\boldsymbol{r}')\Big\}\,,$$

$\theta(E)$ is the step function which is 1 for energies E above threshold and zero otherwise.

4. – Energy behaviour of the reaction amplitude.

We see from (3.9) that in general the amplitude Θ_2 will be different from zero and finite at threshold. It will be in general a slowly varying function of the energy E near threshold. Note that below threshold the condition (3.2) is uniquely substituted with the following:

$$\psi_{2\infty}^{\text{below}}(\boldsymbol{r}) = \theta_2^{\text{below}}(\boldsymbol{k}_1, \boldsymbol{k}_2)\frac{\exp[-k_2 r]}{r}, \tag{4.1}$$

since in this case *there is no arbitrariness for the choice of the contour of integration in the complex plane for the Green's function* $(E + i\varepsilon - H_2^{(0)})^{-1}$. Thus, we get the rule that the substitution

$$|k_2| \to i|k_2| \tag{4.2}$$

brings the physical quantities defined above threshold in the corresponding quantities defined below threshold (*). In particular we will have that:

$$\theta_2^{\text{below}}(\boldsymbol{k}_1, \boldsymbol{k}_2) = \theta_2(\boldsymbol{k}_1, i|k_2|\boldsymbol{n}_2); \tag{4.3}$$

$\boldsymbol{n}_2$ is the unit vector in the direction $\boldsymbol{k}_2$.

Clearly, the expression (4.1) represents the physical condition that below threshold the probability of finding asymptotically the channel 2 is zero.

The reaction cross-section $\mathrm{d}\sigma_2/\mathrm{d}\Omega_2$ is zero at threshold due to the velocity factor v_2 present in (3.4). The derivative of the production cross-section with respect to the energy E will in general be infinite at threshold since $\mathrm{d}v_2/\mathrm{d}E$ is infinite.

5. – Energy behaviour of the scattering amplitude.

We emphasize first of all that in formula (3.8) the second integral (principal value integral) will include some discrete terms whenever the operator $H_2^{(0)} + V_2$ exhibits discrete eigenvalues, which correspond to real bound states in channel 2 in the absence of interaction with channel 1.

We will therefore have resonances when the energy of the incoming beam (smaller than the threshold energy) coincides with the relativistic energy of

(*) The simplest justification to the rule (4.2) one could find in the literature, is that given by NEWTON (see ref. (4)). As a matter of fact, other people seem not to realize that the reason, just like that stated here, why (4.2) should be valid is extremely simple.

the bound states. For an extensive discussion of this point as well as for the derivation of the Breit-Wigner formula ([11]) from a formalism of the kind used here, the reader is referred to FESHBACH's paper ([10]).

When the energy E increases the principal value integral of formula (3.8) becomes bigger and bigger since the minimum of the difference $(E - E'')$ in the denominator becomes smaller and smaller. The integral reaches its maximum at the threshold for channel 2, *i.e.*, when $E = E_{-}^{\text{thr}}$.

The same argument used below threshold can be used above. In this case, however, the energy E falls in the continuum of the energies E''. Then the integral becomes bigger and bigger when E approaches the threshold from above because the cancellation given by the principal value operation becomes less and less effective. Thus, also from above the integral reaches the maximum at threshold, *i.e.*, when $E = E_{+}^{\text{thr}}$.

Because of the other terms present in formula (3.8), we cannot say whether the scattering amplitude itself has a maximum at threshold and in fact in general this will not be the case. Anyway, if we restrict ourselves to energies below threshold, we can easily see that the scattering amplitude will reach a maximum when the threshold is approached from below if the diagonal potential V is negative definite or small compared to the potential $V_R^{(1)} - V_1$ which arises from the virtual transitions channel 1 $\to$ channel 2 $\to$ $\to$ channel 1.

We are left with the discussion of the energy derivative of the scattering amplitude Θ_1. In order to study this in detail, we will consider explicitly the case in which the potential V_2 is spherically symmetric.

For this case we can choose for the variables β the square of the total angular momentum $\boldsymbol{J}^2$, the square of the orbital angular momentum $\boldsymbol{L}^2$, the square of the total spin for the channel 2 $\boldsymbol{S}^2$, and the projection J_z of the total angular momentum.

We will have:

$$(5.1)\qquad \varphi_2(\boldsymbol{r}, E'', \beta'') \equiv \varphi_2(\boldsymbol{r}, E'', J, m, l, s) = \frac{1}{r} c_l(E'') G_l(E'', r) \cdot$$

$$\cdot \sum_{m_l m'} C(lsJ; m_l, m - m_l) C(s_1 s_2 s; m', m - m_l - m') \chi_{s_1}^{m'} \chi_{s_2}^{m - m_l - m'} Y_l^{m_l}(\hat{\boldsymbol{r}}),$$

where $\chi_{s_i}^{m_i}$ is the spin eigenfunction for particle i in channel 2. $G_l(E'', r)$ is the regular solution of the radial wave equation defined at the origin so that

$$(5.2)\qquad \lim_{E'' \to 0} \frac{G_l(E'', r)}{k''^{-l} r j_l(k'' r)} = 1 .$$

([11]) G. BREIT and E. P. WIGNER: *Phys. Rev.*, **49**, 519, 642 (1936).

The function $c_l(E'')$ is the appropriate energy dependent factor that assures that φ_2 satisfies the completeness relation

$$(5.3) \qquad \sum_{Jmls} \int \mathrm{d}E'' \varphi_2(\boldsymbol{r}', E'', J, m, l, s)\varphi_2^*(\boldsymbol{r}, E'', J, m, l, s) = \delta^3(\boldsymbol{r} - \boldsymbol{r}') .$$

It is convenient at this point to normalize the energy in such a way that the threshold corresponds to $E = 0$. Then, $c_l(E'')$ is the square root of the derivative of the spectral function ϱ defined by Newton in Section **1** of ref. ([12]). From his expression for ϱ we can immediately deduce the energy behaviour near $E'' = 0$ of $c_l(E'')$ and therefore of the eigenfunctions $\varphi_2(\boldsymbol{r}, E'', J, m, l, s)$ as:

$$(5.4) \qquad \lim_{E''\to 0^+} \varphi_2(\boldsymbol{r}, E'', J, m, l, s) = 0(E''^{\frac{1}{2}(l+\frac{1}{2})}) .$$

With the help of this formula we see that the scattering amplitude Θ_1, and therefore the scattering cross-section, is finite at threshold because the numerator in the principal value integral will have at least a factor $E''^{\frac{1}{2}}$ which renders the integral summable at $E = 0$ (threshold).

In order to consider the energy derivative of this integral, we have to handle the derivative of the principal value of $(E - E'')^{-1}$. This can be done by partial integration. Writing $f(E'')$ for the term $\sum_{Jmls} \varphi_2(\boldsymbol{r}, E'', J, m, l, s) \cdot \varphi_2^*(\boldsymbol{r}', E'', J, m, l, s)$ we get:

$$\int_0^\infty \mathrm{d}E'' \frac{\partial}{\partial E''}\left[\frac{P}{E - E''}\right] f(E'') = -\left[\frac{P}{E - E''} f(E'')\right]_{E''=0}^{E''=\infty} + \int_0^\infty \mathrm{d}E'' \frac{P}{E - E''} \frac{\partial}{\partial E''} f(E'') .$$

The first term is zero for every E owing to the behaviour of the function $f(E'')$ at $E = 0$ and at infinity. So we have:

$$(5.5) \qquad \int_0^\infty \mathrm{d}E'' \frac{\partial}{\partial E}\left[\frac{P}{E - E''}\right] f(E'') = \int_0^\infty \mathrm{d}E'' \frac{P}{E - E''} \frac{\partial}{\partial E''} f(E'') .$$

Analysing the expression (5.5), we see that this integral has an infinite value when E approaches zero both from below and from above because an extra factor $E''^{\frac{1}{2}}$ will appear in the denominator in this case. This factor belongs to the S-wave part of the sum included in $f(E'')$. On the other hand, for $l \neq 0$ the integral does not exhibit any anomaly at threshold. We will see in a moment that from this statement it will follow that the energy derivative

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([12]) R. G. NEWTON: *Phys. Rev.*, **101**, 1588 (1956).

of the scattering amplitude has an anomalous behaviour at threshold for the channel 2 and that the anomaly is given only by the coupling of channel 1 with the S-wave of channel 2.

In order to discuss the energy derivative of the scattering amplitude, we have to take into explicit consideration the energy derivative of the wave function ψ_1. By taking the energy derivative of equation (2.6a) and retaining the terms in $\partial\psi_1/\partial E$ and the integral which, from the argument given above, exhibits an infinity, as $E \to 0$, we get:

$$(5.6) \qquad [E - H_1^{(0)} - V_R^{(1)} + iV_I^{(1)}\theta(E)] \frac{\partial\psi_1}{\partial E} =$$

$$= V_{12} \sum_{Jmls} \int \mathrm{d}E'' \frac{P}{E - E''} \frac{\partial}{\partial E''} [\varphi_2(E''Jmls)\varphi_2^*(E''Jmls)] V_2\psi_1 ,$$

where we have written explicitly the step function in the V_I term.

The integral equation equivalent to the equation (5.6) will consist of an homogeneous term plus a particular solution of (5.6). Neglecting the homogeneous term which does not contain any infinity as $E \to 0$, we get at last:

$$(5.7) \qquad \frac{\partial\psi_1}{\partial E} = \mathcal{G}(E, V_R, V_I) V_{12} \sum_{Jmls} \int \mathrm{d}E'' \frac{P}{E - E''} \frac{\partial}{\partial E''} [\varphi_2(E''Jmls)\varphi_2^*(E''Jmls)] V_{21}\psi_1 .$$

We do not need to specify the boundary conditions on the Green's function $\mathcal{G}(E, V_R, V_I)$ since our considerations will be independent on their choice.

Now we can take the energy derivative of the scattering amplitude Θ_1. Writing it in symbolic form for the sake of clarity, and retaining only the integral which exhibits an infinity in the limit $E \to 0$, we obtain:

$$(5.8) \qquad \frac{\partial\Theta_1(\boldsymbol{k}_1, \boldsymbol{k}_2')}{\partial E} = -\left(\frac{2\pi}{\hbar}\right)^{\frac{1}{2}} \mu_1 \langle \hbar\boldsymbol{k}_1' | [(V_R^{(1)} - iV_I^{(1)})\mathcal{G} + 1] \cdot$$

$$\cdot \sum_{Jmls} \int \mathrm{d}E'' V_{12} \frac{P}{E - E''} \frac{\partial}{\partial E''} [\varphi_2(E''Jmls)\varphi_2^*(E''Jmls)] V_{21} | \psi_1 \rangle .$$

We remind that E, in our convention, is zero at threshold. We note that, since we are interested only in the left and right derivative of the scattering amplitude Θ_1 separately, we do not need to consider the term (δ-function) which arises from the derivation of the step function $\theta(E)$.

6. – Conclusions.

The expression (5.8) is infinite at the threshold for channel 2 both from above and from below. As we have already seen analysing expression (5.5), this anomaly is due to the presence of a factor $E''^{\frac{1}{2}}$ in the denominator of

the term included in $f(E'')$ which couples channel 1 with the S-wave of channel 2. The coupling with the $l \neq 0$ waves in channel 2 does not give any anomaly in the scattering amplitude and therefore in the scattering cross-section. We then get the result that *the scattering cross-section assumes the form of a cusp or rounded step at the threshold for a new channel. By making a partial wave analysis of the scattered wave, we will find the anomaly in one or more waves with specified orbital angular momentum for channel* 1, *depending on the couplings of channel* 1 *with the S-wave of channel* 2 *allowed by conservation of total angular momentum and conservation of parity, if any.*

It is clear from the above statement how one can get, from phenomena of this kind, information about the spin and parity of the particles involved in the reaction.

It should be noted that the behaviour of the scattering cross-section will be different below and above threshold. In fact in the former case $V_I\theta(E) = 0$ and in the latter $V_I\theta(E) = V_I \neq 0$.

From the preceeding we see that *the anomaly finds its origin exclusively in the presence of the potential* $V_R^{(1)} - V_1$ *in equation* (2.7), *i.e. the potential which arises from the virtual transitions*: *channel* $1 \to$ *channel* $2 \to$ *channel* 1. The energy derivative of this potential is in fact infinite at threshold. That is, responsible for the infinity in $\partial\psi_1/\partial E$; both these anomalies yield the infinity of the energy derivative of the scattering amplitude and of the scattering cross-section. We re-emphasize that, of these transitions, only the one which couples the S-wave of channel 2 is effective in producing the cusp or rounded step in the scattering cross-section.

With the present formalism a very simple interpretation is then given to this characteristic behaviour of the scattering cross-section at the opening up of a new channel. In particular, an explanation is found for the infinity in the energy derivative of the scattering cross-section as the threshold is approached from below. As we have seen the origin of this fact is purely quantum mechanical in contrast to the explanation for the same phenomenon from above the threshold, which can be given on the classical basis of conservation of total flux.

Concluding we want to point out that experimentally the cusp will be the better detectable the greater will be the monochromaticity (better, the energy resolution) of the incoming beam. It is clear that, if the initial wave packet has a large spread in energy, the cusp itself will be washed out.

* * *

The author would like to express his sincere appreciation to Professor R. Newton for many stimulating discussions and useful criticism.

RIASSUNTO

Questo lavoro presenta un nuovo metodo per lo studio delle anomalie che appaiono nelle sezioni d'urto di scattering e di reazione alla soglia per la produzione di un nuovo canale. Con tale metodo è possibile effettivamente dare una spiegazione del perchè la derivata rispetto all'energia della sezione d'urto di scattering è infinita quando la soglia è raggiunta come processo limite per energie minori dell'energia di soglia. L'effetto è puramente quanto-meccanico.

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Der « γ-α » Photoeffekt bei Ba, Ce, Nd und Sm.

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(ricevuto il 29 Maggio 1959)

Zusammenfassung. — Es wurde durch Ausmessen von Spuren in Photoplatten, der « γ-α » Photoeffekt bei Ba, Ce, Nd und Sm untersucht.

Samarium und Neodym sind schwache natürliche α-Strahler und es schien von Interesse diese Elemente sowie die benachbarten, nämlich Cerium und Barium auf einen « γ-α » Photoeffekt zu untersuchen, da gegebenenfalls der Einfluß der Restkerne mit 82 Neutronen, besonders zum Ausdruck kommen könnte

Die γ-Strahlen mit Energien bis zu 30 MeV wurden von dem Betatron des Instituts erzeugt und es fielen $(7 \div 10)$ Rg senkrecht auf Ilford E1/25 μm Platten, die mit je 2 mg der Nitrate der angeführten Elemente beschickt waren; auf den Quadratcentimeter kamen also rund etwas über 10^{17} Kerne der Elemente mit denen die Platten beschickt waren, also rund ein Kern auf tausend Kerne der Atome der Emulsion. Daneben wurden unbeschickte Platten zur Kontrolle bestrahlt und auch beschickte aber nicht bestrahlte Platten nach einigen Tagen entwickelt, um eventuelle natürliche α-Aktivitäten feststellen zu können.

Die Platten wurden mit einem geeichten Meßmikroskop mit numerischer Apertur von 0.8 und einer Gesamtvergrößerung von ca $700\times$ ausgemessen. Es wurden die Längen der Spuren der α-Teilchen in ihren Projektionen auf die Plattenebene, sowie die Tiefendifferenz des Anfangs und des Endes der Spuren bestimmt. Hieraus ergab sich die Länge und der Neigungswinkel ϑ der Spuren.

Das Schrumpfen der Plattendicken beim Entwickeln wurde aus dem Abweichen der Energien der natürlichen α-Strahler von Samarium und Thorium (aus den Versuchsreihen mit dem Samarium) in einem E-ϑ Diagramm bestimmt und im Winkelmaßstab der Diagramme berücksichtigt. Eine Kontrolle der Meßgenauigkeit erfolgte durch Ausmessen von 150 Spuren von α-Teilchen von natürlichem Uran in der Plattenebene; das Sigma der Gauß'schen Verteilungen der Messungen für die Energien 4.18 und 4.76 MeV ergab sich hiebei mit 0.185 MeV.

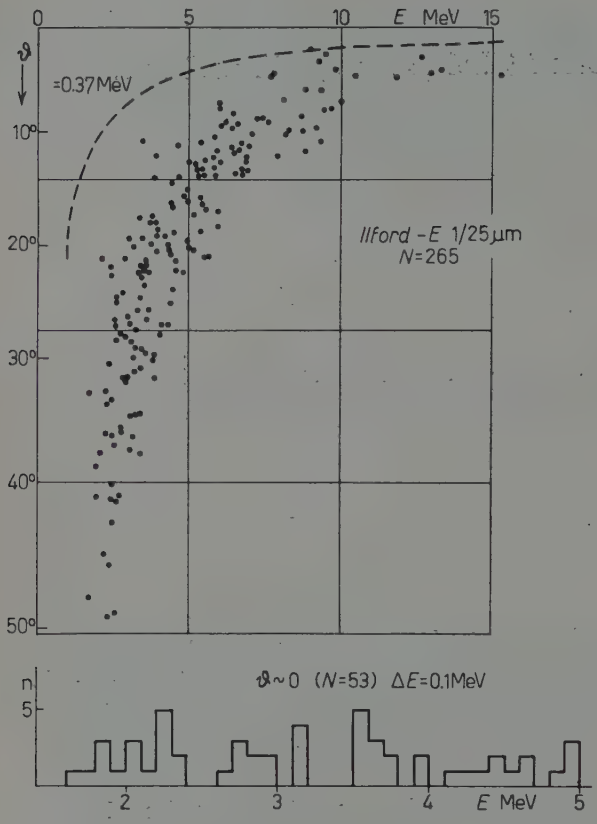

Abb. 1.

Beim «γ-α» Effekt konnten aus den verschiedenen Dicken der Spuren an deren Enden nur Orientierungen in Richtung der γ-Strahlung festgestellt werden.

Die Messungen sind in Diagrammen dargestellt, bei denen gesondert der Effekt für eine Orientierung der Spuren in der Plattenebene (ϑ gleich Null Grad; die Spuren stehen in diesem Fall senkrecht zur Richtung der γ-Strahlen) als E-n Histogramm ($\Delta E = 0.1$ MeV) und für die räumlich orientierten Spuren als Punktdiagramm in E-ϑ Koordinaten, gegeben ist. N ist die Zahl der Messungen.

Abb. 1. gilt für die bestrahlte aber unbeschickte Platte. Die gestrichelte Hyperbel gibt für ein Sigma von 0.37 MeV die Grenze des blinden Teils des Meßbereichs für den die räumliche Orientierung nur sehr schlecht festgestellt werden kann. (Eigenschaften des optischen Systems und der Akkomodation des Messenden).

Abb. 2. gibt die Messungen für Barium, Abb. 3. für Cerium, Abb. 4. für Neodym und Abb. 5 für Samarium. Die Brüche in den Klammern geben an, welcher Teil der Punkte dem untersuchten Element zugeschrieben wurde mit

dem jeweils die Platten beschickt wurden; das Komplement auf « eins » ist dann der Anteil der Emulsion der Platte.

Vergleicht man die Spurenzahlen für die unbeschickten Platten mit den Spurenzahlen der beschickten Platten, so ergeben sich bei Ba, Ce und insbesonders bei Nd um Zehnerpotenzen größere Wirkungsquerschnitte als bei Kernen der unberschickten Platten. Beim Sm bei dem es nicht zu Restkernen mit Neutronenzahlen 82 kommen kann, ändert sich die Größenordnung des Wirkungsquerschnittes wesentlich weniger.

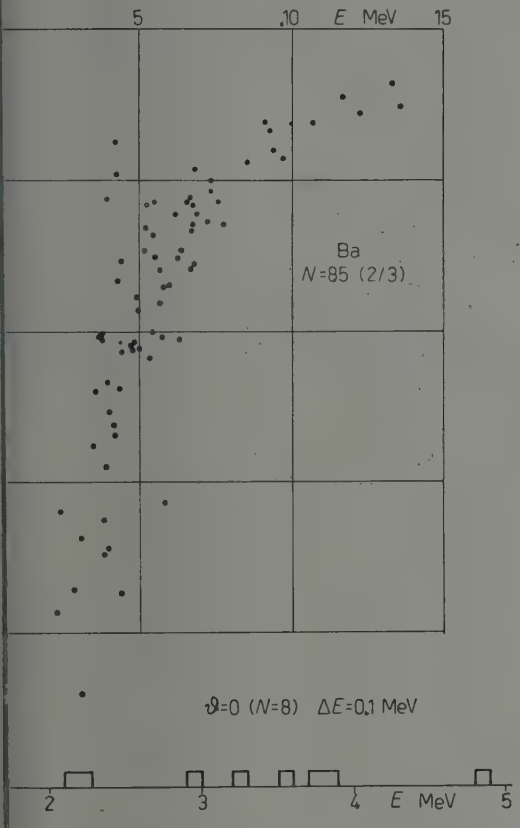

Abb. 2.

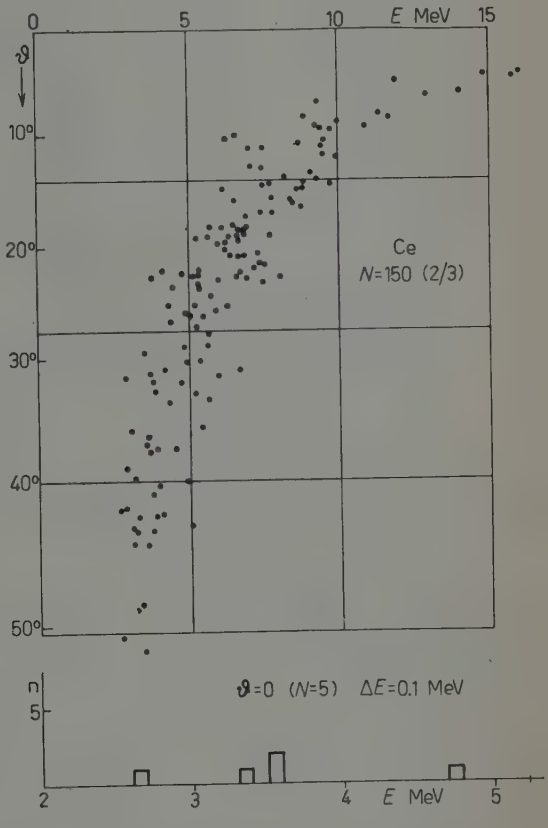

Abb. 3.

Bei Barium und Cerium ist der Photoeffekt praktisch auf Orientierungen mit Winkeln ϑ die größer als Null Grad sind beschränkt. Recht deutlich heben sich Gebiete mit gehäuften Punktezahlen heraus und daneben wieder Gebiete mit sehr geringen Punktezahlen.

Abweichend zeigt sich im Fall von Neodym ein ganzes Spektrum von Energien für einen Winkel ϑ von Null Grad, (Dipolwirkung) neben einer räum-

lichen Verteilung, die etwas von der bei Barium und Cerium abweicht. Bei der räumlichen Verteilung, heben sich auch hier wieder deutlich Häufungsgebiete der gemessenen Punkte heraus. Da dieser Effekt zu unerwartet war, wurden die Messungen mit einer zweiten, gesondert beschickten und be-

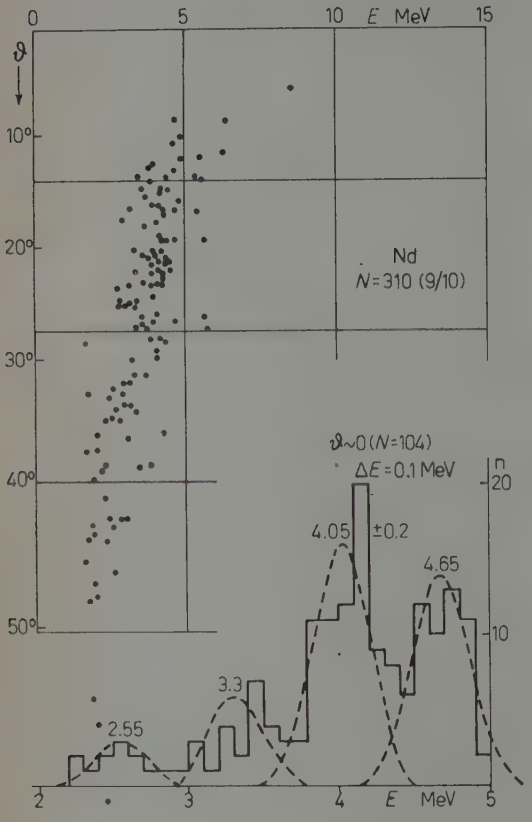

Abb. 4.

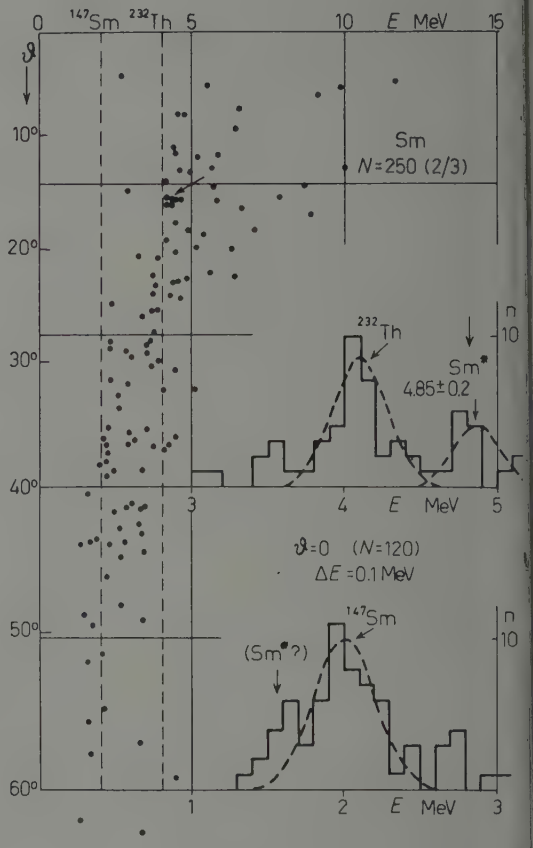

Abb. 5.

strahlten Platte wiederholt; beide Meßreihen sind gemeinsam im Diagramm dargestellt.

Im Fall des Samarium Abb. 5. konnte aus einem Vergleich der reduzierten Statistiken der beschickten und bestrahlten Platten, der beschickten und nicht bestrahlten Platten, sowie der unbeschickten aber bestrahlten Platten mit einer Sicherheit von mehr als 99.73% (entsprechend 3 σ) auf einen Effekt im Gebiet von 3.5 bis 5 MeV geschlossen werden. Die für diese Winkel bezw. Energien in Frage kommenden Stellen wurden im Diagramm hier durch dicke

Pfeile angezeichnet. (Zur Kontrolle wurde aus der Statistik eine genäherte Bestimmung der natürlichen α Aktivität des Samariums gemacht, die den brauchbaren Wert $(1.8 \pm 0.2) \cdot 10^{11}$ Jahre ergab.)

Es ist beabsichtigt diese Untersuchungen mit einem besonderen Alphaspektrogoniometer auf ihre Richtigkeit genauer zu überprüfen.

* * *

Die Messungen sowie die Herstellung der Nitrate wurden von Fräulein M. MODESTO ausgefü rt.

RIASSUNTO (*)

Misurando le tracce su lastre fotografiche è stato esaminato il fotoeffetto « γ-α » in Ba, Ce, Nd e Sm. Oltre alla determinazione degli spettri si ottennero per le reazioni sezioni d'urto sorprendentemente elevate.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XIII, N. 5 1° Settembre 1959

Generalized Stokes Parameters for Waves with Arbitrary Form.

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(ricevuto il 29 Maggio 1959)

Summary. — The correlation matrix of an arbitrary stationary, quasi-monochromatic electromagnetic radiation field is expanded in terms of nine linearly independent matrices and it is shown that the expansion coefficients play a role analogous to the customary Stokes parameters.

1. – Introduction.

Stokes parameters have long been used successfully to characteristize the state of polarization of a quasi-monochromatic plane light wave (*). They are also of interest in the theory of partial coherence (1).

It may seem desirable to find a generalization of the concept and method of these Stokes parameters for fields which are not plane waves. It is not a straightforward problem to find a reasonable generalization. The clue to the solution is, however, supplied by the observation of Fano (2), according to which the Stokes parameters for plane waves may be considered as the expansion coefficients of the 2×2 correlation tensor, if it is expanded in terms of the Pauli algebra. The correlation tensor $\mathscr{E}$ may be defined as (3)

$$\mathscr{E}_{ik}(\boldsymbol{x}) = \langle E_i(\boldsymbol{x}, t)\, E_k^*(\boldsymbol{x}, t)\rangle\,, \qquad (i, k = 1, 2) \tag{1}$$

(*) For a summary see, for example, M. J. Walker: *Amer. Journ. Phys.*, **22**, 170 (1954); W. H. McMaster: *Amer. Journ. Phys.*, **22**, 351 (1954).

(1) E. Wolf: *Nuovo Cimento*, **12**, 1 (1954), or S. Pancharatnam: *Proc. Ind. Acad. Sci.*, A **44**, 398 (1957).

(2) U. Fano: *Phys. Rev.*, **93**, 121 (1954).

(3) E. Wolf: *Nuovo Cimento*, **12**, 1 (1954).

where E_k is the complex electric field,

$$E_k = E_k^{(r)} + iE_k^{(i)} . \tag{2}$$

Here $E_k^{(r)}$ is the physical field, and $E_k^{(i)}$ is the Hilbert transform of $E_k^{(r)}$. Putting

$$E_k(\boldsymbol{x}, t) \equiv A_k(\boldsymbol{x}, t) \exp[i\psi_k(\boldsymbol{x}, t)] , \tag{3}$$

where A_k and ψ_k are real, the physical field becomes

$$E_k^{(r)}(\boldsymbol{x}, t) = A_k(\boldsymbol{x}, t) \cos \psi_k(\boldsymbol{x}, t) . \tag{4}$$

Defining the phase factor Φ by

$$\Phi_k = \psi_k + \bar{\omega}t , \tag{5}$$

where $\bar{\omega}$ is the average frequency, we may write

$$E_k^{(r)} = A_k \cos(\Phi_k - \bar{\omega}t) . \tag{6}$$

The correlation matrix reads

$$\mathscr{E}(\boldsymbol{x}) = \begin{pmatrix} \langle A_1^2 \rangle & \langle A_1 A_2 \exp[i(\Phi_1 - \Phi_2)] \rangle \\ \langle A_1 A_2 \exp[-i(\Phi_1 - \Phi_2)] \rangle & \langle A_2^2 \rangle \end{pmatrix} . \tag{7}$$

If we put

$$\mathscr{E} = \tfrac{1}{2} \sum_{\alpha=0}^{3} s_\alpha \sigma_\alpha , \tag{8}$$

where σ_α are the elements of the Pauli algebra, *i.e.*

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

we find

$$\begin{cases} s_0 = \langle A_1^2 \rangle + \langle A_2^2 \rangle , \\ s_1 = \langle A_1^2 \rangle - \langle A_2^2 \rangle , \\ s_2 = 2\langle A_1 A_2 \cos(\Phi_1 - \Phi_2) \rangle , \\ s_3 = 2\langle A_1 A_2 \sin(\Phi_1 - \Phi_2) \rangle . \end{cases} \tag{9}$$

These expressions have indeed the form of the customary definitions of the

Stokes parameters. In terms of these quantities the correlation matrix (7) becomes

$$\mathscr{E}=\frac{1}{2}\begin{pmatrix} s_0+s_1 & s_2-is_3 \\ s_2+is_3 & s_0-s_1 \end{pmatrix}. \tag{7a}$$

2. – The generalized stokes parameters.

Let us now turn to the general case of a stationary quasi-monochromatic field which is not a plane wave. In this case the correlation matrix will be a 3×3 matrix; the elements of which are again given by (1), where now $i, k = 1, 2, 3$. By analogy to (7) we may then write

$$\mathscr{E}(\boldsymbol{x}) = \tag{10}$$

$$= \begin{pmatrix} \langle A_1^2\rangle & \langle A_1A_2\exp[i(\Phi_1-\Phi_2)]\rangle & \langle A_1A_3\exp[i(\Phi_1-\Phi_3)]\rangle \\ \langle A_1A_2\exp[-i(\Phi_1-\Phi_2)]\rangle & \langle A_2^2\rangle & \langle A_2A_3\exp[i(\Phi_2-\Phi_3)]\rangle \\ \langle A_1A_3\exp[-i(\Phi_1-\Phi_3)]\rangle & \langle A_2A_3\exp[-i(\Phi_2-\Phi_3)]\rangle & \langle A_3^2\rangle \end{pmatrix}.$$

We note that (10) (as well as (7)) is Hermitian.

In analogy to the procedure of equation (8), we now want to expand (10) in terms of a minimal set of linearly independent 3×3 matrices. The appropriate algebra which enables us to expand any 3×3 matrix in terms of nine linearly independent matrices may be deduced from the « three dimensional » Kemmer algebra, and will be discussed elsewhere [4]. Here we merely list the nine matrices, in a convenient Hermitian representation:

$$\left\{\begin{array}{l} \varrho_0=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \quad \varrho_1=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}, \quad \varrho_2=\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}, \\ \varrho_3=\begin{pmatrix}0&i&0\\-1&0&i\\0&-i&0\end{pmatrix}, \quad \varrho_4=\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}, \quad \varrho_5=\begin{pmatrix}0&0&0\\0&0&-1\\0&-1&0\end{pmatrix}, \\ \varrho_6=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}, \quad \varrho_7=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}, \quad \varrho_8=\begin{pmatrix}0&0&i\\0&0&0\\-i&0&0\end{pmatrix}. \end{array}\right. \tag{11}$$

[4] P. ROMAN: *Proc. Phys. Soc.*, to be published.

If we set

$$\mathscr{E} = \tfrac{1}{2}\sum_{\alpha=0}^{8} r_\alpha \varrho_\alpha , \tag{12}$$

we find for the expansion coefficients:

$$\begin{cases}
r_0 = 2\langle A_2^2\rangle = 2\mathscr{E}_{22} , \\
r_1 = \langle A_1^2\rangle - \langle A_3^2\rangle = \mathscr{E}_{11} - \mathscr{E}_{33} , \\
r_2 = 2\langle A_1 A_2 \cos(\Phi_1 - \Phi_2)\rangle = 2\,\mathrm{Re}\,\mathscr{E}_{12} , \\
r_3 = 2\langle A_1 A_2 \sin(\Phi_1 - \Phi_2)\rangle = -2\,\mathrm{Im}\,\mathscr{E}_{21} , \\
r_4 = \langle A_1^2\rangle + \langle A_3^2\rangle - 2\langle A_2^2\rangle = \mathscr{E}_{11} + \mathscr{E}_{33} - 2\mathscr{E}_{22} , \\
r_5 = 2\langle A_1 A_2 \cos(\Phi_1 - \Phi_2)\rangle - 2\langle A_2 A_3 \cos(\Phi_2 - \Phi_3)\rangle = \\
\qquad\qquad = 2\,\mathrm{Re}\,\mathscr{E}_{12} - 2\,\mathrm{Re}\,\mathscr{E}_{23} , \\
r_6 = 2\langle A_1 A_2 \sin(\Phi_1 - \Phi_2)\rangle - 2\langle A_2 A_3 \sin(\Phi_2 - \Phi_3)\rangle = \\
\qquad\qquad = -2\,\mathrm{Im}\,\mathscr{E}_{21} + 2\,\mathrm{Im}\,\mathscr{E}_{32} , \\
r_7 = 2\langle A_1 A_3 \cos(\Phi_1 - \Phi_3)\rangle = 2\,\mathrm{Re}\,\mathscr{E}_{13} , \\
r_8 = 2\langle A_1 A_3 \sin(\Phi_1 - \Phi_3)\rangle = -2\,\mathrm{Im}\,\mathscr{E}_{31} .
\end{cases} \tag{13}$$

The nine expansion coefficients r_α we shall call generalized Stokes parameters. In terms of these parameters the correlation matrix (10) can be written as follows:

$$\mathscr{E} = \frac{1}{2}\begin{pmatrix} r_0 + r_1 + r_4 & r_2 + ir_3 & r_7 + ir_8 \\ r_2 - ir_3 & r_0 & r_2 - r_5 + ir_3 - ir_6 \\ r_7 - ir_8 & r_2 - r_5 - ir_3 + ir_6 & r_0 - r_1 + r_4 \end{pmatrix} . \tag{14}$$

We note that the intensity of the radiation, $I = \langle A_1^2 + A_2^2 + A_3^2\rangle$, can be expressed as

$$I = \tfrac{3}{2} r_0 + r_4 . \tag{15}$$

In order to see the significance of the parameters r_α, let us consider some special cases.

Suppose that, at a certain point, $\langle A_3^2\rangle = 0$. Then it follows that $A_3 = 0$ and $E_3 = 0$ at any time, so that at this point the general field degenerates into a « plane wave » which travels in the Z-direction. We then have, from (13), that

$$r_0 = 2\langle A_2^2\rangle , \quad r_1 = \langle A_1^2\rangle , \quad r_2 = r_5 = 2\langle A_1 A_2 \cos(\Phi_1 - \Phi_2)\rangle ,$$
$$r_3 = r_6 = 2\langle A_1 A_2 \sin(\Phi_1 - \Phi_2)\rangle , \quad r_4 = \langle A_1^2\rangle - 2\langle A_2^2\rangle ,$$
$$r_7 = r_8 = 0 .$$

We see that there are now only four independent parameters. Introducing the linear combinations

$$(16)\qquad \begin{cases} r_1 + \tfrac{1}{2} r_0 \equiv r_0' = \langle A_1^2 \rangle + \langle A_2^2 \rangle , \\ r_1 - \tfrac{1}{2} r_0 \equiv r_1' = \langle A_1^2 \rangle - \langle A_2^2 \rangle , \\ r_2 \equiv r_2' = 2\langle A_1 A_2 \cos(\Phi_1 - \Phi_2) \rangle , \\ r_3 \equiv r_3' = 2\langle A_1 A_2 \sin(\Phi_1 - \Phi_2) \rangle , \end{cases}$$

we observe that these correspond to the four familiar Stokes parameters (9) characterizing the polarization of a beam propagated in the Z-direction. A similar simplification occurs if $E_1 = 0$ or if $E_2 = 0$.

3. – Unpolarized radiation.

Let us consider a field for which

$$\langle A_1^2 \rangle = \langle A_2^2 \rangle , \qquad \langle A_3^2 \rangle = \text{arbitrary} ,$$

at a certain point, and all other elements of the correlation matrix (10) vanish. We shall say that (at this particular point) the field is « completely unpolarized in the XY-plane ». From (13) it then follows that there are only two independent non-vanishing parameters, *i.e.* r_0 and r_4, and that $r_0 = -r_4$. The correlation matrix therefore has the form

$$(17a)\qquad \bar{\mathscr{E}}^{XY} = \frac{1}{2}\begin{pmatrix} \bar{r}_0 & 0 & 0 \\ 0 & \bar{r}_0 & 0 \\ 0 & 0 & \bar{r}_0 + 2\bar{r}_4 \end{pmatrix} .$$

(To avoid confusion, we have used a bar over both the correlation matrix and its elements; the superscript «XY» refers to the fact that the radiation is unpolarized in this plane).

Similarly, if the field is « completely unpolarized in the XZ-plane », *i.e.* if

$$\langle A_1^2 \rangle = \langle A_3^2 \rangle , \qquad \langle A_2^2 \rangle = \text{arbitrary} ,$$

and all other correlation matrix elements vanish, then we obtain

$$(17b)\qquad \bar{\mathscr{E}}^{XZ} = \frac{1}{2}\begin{pmatrix} \bar{r}_0 + \bar{r}_4 & 0 & 0 \\ 0 & \bar{r}_0 & 0 \\ 0 & 0 & \bar{r}_0 + \bar{r}_4 \end{pmatrix} .$$

SOCIETÀ ITALIANA DI FISICA

IL NUOVO CIMENTO

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Colloque International sur les Ultrasons (Rome, 1950).
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Congrès International sur les particules instables lourdes et sur les événements de haute énergie dans les Rayons Cosmiques (Padoue, 1954).
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MORPURGO et FRANZINETTI. *An Introduction to the Physics of the New Particles* (1957).

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34, Via Irnerio - BOLOGNA

Finally, for a field « completely unpolarized in the YZ-plane », *i.e.* for

$$\langle A_2^2 \rangle = \langle A_3^2 \rangle , \quad \langle A_1^2 \rangle = \text{arbitrary} ,$$

and all other matrix elements zero, we find

$$\bar{\mathscr{E}}^{YZ} = \frac{1}{2} \begin{pmatrix} \bar{r}_0 + 2\bar{r}_4 & 0 & 0 \\ 0 & \bar{r}_0 & 0 \\ 0 & 0 & \bar{r}_0 \end{pmatrix} . \tag{17c}$$

Let us further specify the situation by assuming that, in addition to the above specifications, we have also $\langle A_3^2 \rangle = 0$, or $\langle A_2^2 \rangle = 0$, or $\langle A_1^2 \rangle = 0$. These conditions imply $E_3 = 0$, $E_2 = 0$, or $E_1 = 0$ respectively, *i.e.* the flow at any time will be perpendicular to the plane which is considered. Then, in each case, we are left with just one non-vanishing parameter, *viz*, $\bar{r}_0$, $\bar{r}_4$, or $\bar{r}_0$ respectively and we find

$$\bar{\bar{\mathscr{E}}}^{XY} = \frac{1}{2} \begin{pmatrix} \bar{\bar{r}}_0 & 0 & 0 \\ 0 & \bar{\bar{r}}_0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{18a}$$

$$\bar{\bar{\mathscr{E}}}^{XZ} = \frac{1}{2} \begin{pmatrix} \bar{r}_4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \bar{\bar{r}}_4 \end{pmatrix} , \tag{18b}$$

$$\bar{\bar{\mathscr{E}}}^{YZ} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \bar{\bar{r}}_0 & 0 \\ 0 & 0 & \bar{\bar{r}}_0 \end{pmatrix} . \tag{18c}$$

(The double-bars refer to this further specialization that we have made.) We notice that in these cases the intensity becomes $\bar{r}_0$, $\bar{r}_4$ and $\bar{r}_0$ respectively.

Consider now the case for which, at a certain point, the light is unpolarized in the plane perpendicular to the direction of the average flow. By this we mean that if we choose an orthogonal local co-ordinate system ξ, η, ζ such that ζ points in the direction of the average flow and ξ and η are in the plane perpendicular to ζ, then

$$\langle A_\xi^2 \rangle = \langle A_\eta^2 \rangle , \quad \langle A_\zeta^2 \rangle = \text{arbitrary} ,$$

and all other matrix elements of $\mathscr{E}$ with respect to the above-specified co-

ordinate system vanish. Let us find the corresponding form of the correlation matrix (at that point) in an arbitrarily specified co-ordinate system X, Y, Z.

Suppose the direction of the average flow at the point under consideration, *i.e.* the vector ζ, is made to coincide with the Z-axis, and ξ and η with the directions of the X and Y axes, if we perform a rotation by means of the orthogonal matrix $M^{-1} = (m_{ik})$. The condition for orthogonality is $\widetilde{M}M = 1$, where the tilda means transpose. If the light is completely unpolarized then, after the rotation M^{-1}, the correlation matrix $\overline{\mathscr{E}}$ must assume the form (17*a*). Consequently, the form of $\overline{\mathscr{E}}$ in the arbitrary system XYZ will be obtained from (17*a*) by the orthogonal similarity-transformation

$$\overline{\mathscr{E}} = M^{-1}\overline{\mathscr{E}}^{XY}M .$$

For the elements of $\overline{\mathscr{E}}$ we thus find

$$\overline{\mathscr{E}}_{ik} = M^{-1}_{ir}\,\overline{\mathscr{E}}^{XY}_{rs}\,M_{sk} = m_{ri}\,\overline{\mathscr{E}}^{XY}_{rs}\,m_{sk} ,$$

and, since only the three diagonal elements of $\overline{\mathscr{E}}^{XY}$ are different from zero, we obtain

$$\overline{\mathscr{E}}_{ik} = \tfrac{1}{2}\left[m_{1i}m_{1k}\overline{r}_0 + m_{2i}m_{2k}\overline{r}_0 + m_{3i}m_{3k}(\overline{r}_0 + 2\overline{r}_4)\right] . \tag{19}$$

This shows that the correlation matrix of completely unpolarized radiation is symmetric in any co-ordinate system. In view of (14) this means that $r_3 = r_6 = r_8 = 0$, so that $\overline{\mathscr{E}}$ will be real and has the form

$$\overline{\mathscr{E}} = \frac{1}{2}\begin{pmatrix} r_0 + r_1 + r_4 & r_2 & r_7 \\ r_2 & r_0 & r_2 - r_5 \\ r_7 & r_2 - r_5 & r_0 - r_1 + r_4 \end{pmatrix} . \tag{20}$$

Naturally, this is true only for those points at which the field is unpolarized.

As a matter of fact, we would have obtained the same result if, in the course of the calculation, we had rotated the co-ordinate system in such a way that the direction of the average flow coincided with the Y or the X direction.

If, in addition, the field is not only completely polarized in the plane perpendicular to the average flow, but also $\langle A_\zeta^2 \rangle = 0$, *i.e.* $E_\zeta = 0$ at any time, then after the appropriate rotation the correlation matrix must assume the form (18*a*). This then gives, instead of (19),

$$\overline{\mathscr{E}}_{ik} = \tfrac{1}{2}(m_{1i}m_{1k} + m_{2i}m_{2k})\overline{r}_0 . \tag{21}$$

The form of $\overline{\mathscr{E}}$ will be similar to that of (20); but, as one finds from (21), the discriminant of $\overline{\mathscr{E}}$ must vanish. (This follows immediately, without any calculation, from the fact that (18*a*) obviously has a vanishing discriminant; and discriminants are invariant under rotations.) Consequently there will be a relationship between the parameters r_0, r_1, r_2, r_4, r_5, and r_7; and we may express r_7 as a (quadratic) function of the others.

We remark further that, as seen from (21), in the present case the elements of the correlation matrix are completely determined by the direction of the average flow and by the intensity. (The intensity in our particular co-ordinate system is $\bar{r}_0$, and thus, being the trace of a matrix, is an invariant.)

4. - Quadratic relations between the generalized Stokes parameters.

Let us restrict ourselves for the moment to the case for which the field is strictly monochromatic and hence «*completely polarized*». In that case the A_k and the Φ_k, defined by (3), (4) and (5), become time-independent. Consequently we can omit the symbols $\langle ... \rangle$ of averaging in the expressions (13). Then we see that by introducing the generalized Stokes parameters through (13), we have expressed the six quantities A_1, A_2, A_3, $\Phi_1 - \Phi_2$, $\Phi_2 - \Phi_3$, $\Phi_1 - \Phi_3$ by the nine quantities $r_\varkappa$. Therefore there must exist three relationships between these nine parameters. Since by their introduction as expansion coefficients with respect to a set of linearly independent matrices, the generalized Stokes parameters must be linearly independent, the relations between them must be (at least) quadratic. By systematic combination we find that these three relations are as follows:

$$-r_0^2 + r_0 r_1 + r_2^2 + r_3^2 - r_0 r_4 - 2r_2 r_5 - 2r_3 r_6 + r_5^2 + r_6^2 = 0\,, \tag{22}$$

$$-r_0^2 + r_1^2 - 2r_0 r_4 - r_4^2 + r_7^2 + r_8^2 = 0\,, \tag{23}$$

$$r_0^2 + r_0 r_1 + r_0 r_4 - r_2^2 - r_3^2 = 0\,. \tag{24}$$

They are the generalization of the well-known relation $s_1^2 + s_2^2 + s_3^2 - s_0^2 = 0$, which relates the ordinary Stokes parameters of strictly monochromatic (and hence completely polarized) plane waves.

If the light is not strictly monochromatic, then the relations (22), (23), (24) of course become invalid. It is, however, important to note that the left hand sides of these equations are nothing but $-4\,\mathrm{Min}\,\mathscr{E}_{11}$, $-4\,\mathrm{Min}\,\mathscr{E}_{22}$, and $-4\,\mathrm{Min}\,\mathscr{E}_{33}$, respectively. This can be seen from (14). Inspection of (10) tells us, if we utilize the Schwartz inequality for integrals, that the minors of the diagonal elements of $\mathscr{E}$ are non-negative. Consequently, in the general case,

the relations (22), (23) and (24) assume the form

$$(22a) \qquad -r_0^2 + r_0 r_1 + r_2^2 + r_3^2 - r_0 r_4 - 2r_2 r_5 - 2r_3 r_6 + r_5^2 + r_6^2 \leqslant 0 ,$$

$$(23a) \qquad -r_0^2 + r_1^2 - 2r_0 r_4 - r_4^2 + r_7^2 + r_8^2 \leqslant 0 ,$$

$$(24a) \qquad r_0^2 + r_0 r_1 + r_0 r_4 - r_2^2 - r_3^2 \leqslant 0 .$$

* * *

The author is obliged to Dr. E. WOLF for calling his attention to the problem and for several enlightening discussions.

RIASSUNTO (*)

Si sviluppa in termini di nove matrici linearmente indipendenti la matrice di correlazione di un arbitrario campo stazionario di radiazione elettromagnetica quasi-monocromatica, e si dimostra che i coefficienti dello sviluppo hanno un ruolo analogo a quello rappresentato dai consueti parametri di Stokes.

(*) *Traduzione a cura della Redazione.*

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Generalization of the Levinson-Jauch Theorem to an Arbitrary Number of Channels.

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(ricevuto il 30 Maggio 1959)

Summary. — For a system having a finite number of channels the bound states of the free and complete Hamiltonian are shown to be simply related to the determinant and trace of the S-matrix.

Recently JAUCH (1) gave a general proof of the relation between bound states and phase shifts originally given by LEVINSON (2) for potential scattering. It will be shown that Jauch's proof is readily generalized to systems having a finite number of channels and to cases where the free and complete Hamiltonian has different one particle stationary states.

We closely follow Jauch's (1) notation throughout. Let H, H_0 be the complete and free Hamiltonians respectively. The outgoing states of H which coincide with the states of ω_q^i of H_0 at $-\infty$ will be designated by Ω_q^i, where i labels the channels, such that

$$H\Omega_q^i = E_q^i \Omega_q^i ,$$

$$H_0 \omega_q^i = \varepsilon_q^i \omega_q^i .$$

The bound states of H and H_0 will be indicated by Ω_α with $\alpha = 1, \ldots, n$, ω_α with $\alpha = 1, \ldots, n_0$ with

$$H\Omega_\alpha = E_\alpha \Omega_\alpha ,$$

$$H_0 \omega_\alpha = \varepsilon_\alpha \omega_\alpha .$$

(*) Supported by the Wisconsin Alumni Research Foundation.

(1) J. M. JAUCH: *Helv. Phys. Acta*, **30**, 143 (1957).

(2) N. LEVINSON: *Kgl. Dan. Vid. Selskab, Math. Fys. Medd.*, **25**, no. 9 (1949).

Finally the stationary states of H which are not stable for H_0 alone will be Ω_q' and the stationary states of H_0 which are not stable for H will be labeled ω_q', such that

$$H\Omega_q' = E_q'\Omega_q' ,$$

$$H_0\omega_q' = \varepsilon_q'\omega_q' .$$

Defining the wave operator as

$$\Omega^+ = \sum_{q,i} \omega_q^i\rangle\langle\Omega_q^i , \tag{1}$$

$$\Omega = \sum_{q,i} \Omega_q^i\rangle\langle\omega_q^i , \tag{2}$$

from Equ. (1) and (2)

$$\Omega^+\Omega = \sum_{q,i} \omega_q^i\rangle\langle\omega_q^i = 1 - \sum_{\alpha=1}^{n_0} \omega_\alpha\rangle\langle\omega_\alpha - \sum_{q=1}^{N_0} \omega_q'\rangle\langle\omega_q' , \tag{3}$$

$$\Omega\Omega^+ = \sum_{q,i} \Omega_q^i\rangle\langle\Omega_q^i = 1 - \sum_{\alpha=1}^{n} \Omega_\alpha\rangle\langle\Omega_\alpha - \sum_{q=1}^{N} \Omega_q'\rangle\langle\Omega_q' . \tag{4}$$

Letting

$$\Omega^+\Omega - \Omega\Omega^+ = \Lambda , \tag{5}$$

it follows that

$$\text{trace } \Lambda = n + N - n_0 - N_0 . \tag{6}$$

Designating Ω_α, Ω_q' by Φ_α and ω_α, ω_q' by φ_α, from Equ. (5)

$$\begin{aligned} \text{trace } \Lambda = \sum_{\alpha,\alpha'} [&\langle\varphi_\alpha|\Omega^+|\varphi_{\alpha'}\rangle\langle\varphi_{\alpha'}|\Omega|\varphi_\alpha\rangle - \langle\varphi_\alpha|\Omega|\varphi_{\alpha'}\rangle\langle\varphi_{\alpha'}|\Omega^+|\varphi_\alpha\rangle] + \\ + \sum_{\alpha,q,i} [&\langle\varphi_\alpha|\Omega^+|\omega_q^i\rangle\langle\omega_q^i|\Omega|\varphi_\alpha\rangle - \langle\varphi_\alpha|\Omega|\omega_q^i\rangle\langle\omega_q^i|\Omega^+|\varphi_\alpha\rangle + \\ + &\langle\omega_q^i|\Omega^+|\varphi_\alpha\rangle\langle\varphi_\alpha|\Omega|\omega_q^i\rangle - \langle\omega_q^i|\Omega|\varphi_\alpha\rangle\langle\varphi_\alpha|\Omega^+|\omega_q^i\rangle] + \\ + \sum_{\substack{q,i \\ q',i'}} [&\langle\omega_q^i|\Omega^+|\omega_{q'}^{i'}\rangle\langle\omega_{q'}^{i'}|\Omega|\omega_q^i\rangle - \langle\omega_q^i|\Omega|\omega_{q'}^{i'}\rangle\langle\omega_{q'}^{i'}|\Omega^+|\omega_q^i\rangle] . \end{aligned} \tag{7}$$

Since $\langle\varphi_\alpha|\Omega|\varphi_{\alpha'}\rangle = 0$ the first term of (7) vanishes. Using Equ. (1) and (2) to evaluate the second sum in Equ. (7) it reduces to

$$\sum_{q,i,\alpha} \langle\varphi_\alpha|\Omega_q^i\rangle\langle\Omega_q^i|\varphi_\alpha\rangle - \sum_{q,i,\alpha} \langle\Omega_q^i|\varphi_\alpha\rangle\langle\varphi_\alpha|\Omega_q^i\rangle = 0 .$$

Thus

$$\text{trace } \Lambda = \sum_{q,i,q',i'} [\langle\omega_q^i|\Omega^+|\omega_{q'}^{i'}\rangle\langle\omega_{q'}^{i'}|\Omega|\omega_q^i\rangle - \langle\omega_q^i|\Omega|\omega_{q'}^{i'}\rangle\langle\omega_{q'}^{i'}|\Omega^+|\omega_q^i\rangle] . \tag{8}$$

Using Equ. (2)

$$\langle \omega_{q'}^{i'} | \Omega | \omega_q^i \rangle = \langle \omega_{q'}^{i'} | \Omega_{q'}^i \rangle = \delta_{qq'} \delta_{ii'} + \frac{\langle \omega_{q'}^{i'} | G | \omega_q^i \rangle}{\varepsilon_q^i - \varepsilon_{q'}^{i'} + i\varepsilon} \quad (3), \tag{9}$$

where G is related to the S-matrix by

$$\langle \omega_{q'}^{i'} | S | \omega_q^i \rangle = \delta_{qq'} \delta_{ii'} - 2\pi i \, \delta(\varepsilon_{q'}^{i'} - \varepsilon_q^i) \langle \omega_{q'}^{i'} | G | \omega_q^i \rangle \, . \tag{10}$$

Since the unitarity of S has its simplest expression in states of definite angular momentum it is convenient to define a wave operator of definite angular momentum L and L_z and in taking the trace of Equ. (5) to use states of the same angular momentum. We shall not use additional subscripts to indicate this change but will use Equ. (6) for states of fixed L, L_z. Further, instead of ω_q^i we shall use $\varphi_i(\varepsilon)$ referring to fixed L, L_z.
Consider

$$\langle \varphi_i(\varepsilon') | I | \varphi_i(\varepsilon) \rangle \equiv \tag{11}$$

$$\equiv \sum_j \int \mathrm{d}\varepsilon'' \left[\langle \varphi_i(\varepsilon') | \Omega^+ | \varphi_j(\varepsilon'') \rangle \langle \varphi_j(\varepsilon'') | \Omega | \varphi_i(\varepsilon) \rangle - \langle \varphi_i(\varepsilon') | \Omega | \varphi_j(\varepsilon'') \rangle \langle \varphi_j(\varepsilon'') | \Omega^+ | \varphi_i(\varepsilon) \rangle \right].$$

Using Equ. (9) in (11)

$$\begin{aligned} \langle \varphi_i(\varepsilon') | I | \varphi_i(\varepsilon) \rangle = {} & \pi^2 \delta(\varepsilon' - \varepsilon) \sum_j \left[\langle \varphi_j(\varepsilon') | G | \varphi_i(\varepsilon') \rangle^* \langle \varphi_j(\varepsilon) | G | \varphi_i(\varepsilon) \rangle - \right. \\ & \left. - \langle \varphi_i(\varepsilon') | G | \varphi_j(\varepsilon') \rangle \langle \varphi_i(\varepsilon) | G | \varphi_j(\varepsilon) \rangle^* \right] + \\ & + \sum_j P \int \frac{\mathrm{d}\varepsilon''}{(\varepsilon'' - \varepsilon')(\varepsilon'' - \varepsilon)} \left[\langle \varphi_j(\varepsilon'') | G | \varphi_i(\varepsilon') \rangle^* \langle \varphi_j(\varepsilon'') | G | \varphi_i(\varepsilon) \rangle - \right. \\ & \left. - \langle \varphi_i(\varepsilon') | G | \varphi_j(\varepsilon'') \rangle \langle \varphi_i(\varepsilon) | G | \varphi_j(\varepsilon'') \rangle^* \right] + \\ & + \frac{i\pi}{\varepsilon - \varepsilon'} \sum_j \left[\langle \varphi_j(\varepsilon') | G | \varphi_i(\varepsilon') \rangle^* \langle \varphi_j(\varepsilon') | G | \varphi_i(\varepsilon) \rangle + \langle \varphi_j(\varepsilon) | G | \varphi_i(\varepsilon') \rangle^* \langle \varphi_j(\varepsilon) | G | \varphi_i(\varepsilon) \rangle - \right. \\ & \left. - \langle \varphi_i(\varepsilon') | G | \varphi_j(\varepsilon') \rangle \langle \varphi_i(\varepsilon) | G | \varphi_j(\varepsilon') \rangle^* - \langle \varphi_i(\varepsilon') | G | \varphi_j(\varepsilon) \rangle \langle \varphi_i(\varepsilon) | G | \varphi_j(\varepsilon) \rangle^* \right]. \end{aligned} \tag{12}$$

Note that from Equ. (8)

$$A = \sum_i \int \langle \varphi_i(\varepsilon) | I | \varphi_i(\varepsilon) \rangle \, \mathrm{d}\varepsilon \, . \tag{13}$$

Thus the first term of Equ. (12) does not contribute to (13). If further

$$P \int \frac{\mathrm{d}\varepsilon''}{(\varepsilon'' - \varepsilon)^2} \sum_{i,j} \left[|\langle \varphi_j(\varepsilon'') | G | \varphi_i(\varepsilon) \rangle|^2 - |\langle \varphi_i(\varepsilon) | G | \varphi_j(\varepsilon'') \rangle|^2 \right],$$

(3) M. GELL-MANN and M. L. GOLDBERGER: *Phys. Rev.*, **91**, 398 (1953).

is uniformly and absolutely convergent with respect to ε it does not contribute to trace Λ since its argument is odd in the interchange of ε and ε''. Note that the principal value is well defined since at $\varepsilon = \varepsilon''$ the argument is zero.

It is convenient to choose a basis that diagonalizes S such that

$$\langle \psi_i(\varepsilon') | S | \psi_j(\varepsilon) \rangle = \delta(\varepsilon - \varepsilon') \delta_{ij} \mathcal{S}_i(\varepsilon) \; ; \qquad \mathcal{S}_i(\varepsilon) = \exp[2i\delta_i(\varepsilon)]. \tag{14}$$

Since $\psi_i(\varepsilon)$ can be obtained by applying a unitary transformation to $\varphi_i(\varepsilon)$, it follows from equation (12) that

$$\text{trace } \Lambda = \sum_{\varepsilon_l} \sum_{i=1}^{l} \int_{\varepsilon_l}^{\varepsilon_{l+1}} i\pi \left[\langle \psi_i(\varepsilon) | G | \psi_i(\varepsilon) \rangle^* \frac{\partial}{\partial \varepsilon} \langle \psi_i(\varepsilon) | G | \psi_i(\varepsilon) \rangle - \right. \tag{15}$$

$$\left. - \frac{\partial}{\partial \varepsilon} \langle \psi_i(\varepsilon) | G | \psi_i(\varepsilon) \rangle^* \langle \psi_i(\varepsilon) | G | \psi_i(\varepsilon) \rangle \right] d\varepsilon \, ,$$

where in the energy interval ε_l to ε_{l+1} only l channels are open. In obtaining Equ. (15) it is further assumed that the various amplitudes $\langle \varphi_j(\varepsilon') | G | \varphi_i(\varepsilon) \rangle$, $i \neq j$, in the neighborhood of ε_l have an energy dependence of $(\varepsilon - \varepsilon_l)^\alpha$, where α is any positive number.

Using Equ. (14) with (10) yields

$$\langle \psi_i(\varepsilon) | G | \psi_i(\varepsilon) \rangle = -\frac{1}{\pi} \exp[i\delta_i(\varepsilon)] \sin \delta_i(\varepsilon) \, . \tag{16}$$

Setting

$$\sigma_i(\varepsilon) = \delta_i(\varepsilon) - \tfrac{1}{2} \sin 2\delta_i(\varepsilon) \tag{17}$$

and using Equ. (6), (15), (16), (17)

$$\pi(n + N - n_0 - N_0) = \sum_{\varepsilon_l} \sum_{i=1}^{l} \left(\sigma_i(\varepsilon_l) - \sigma_i(\varepsilon_{l+1}) \right) . \tag{18}$$

Note that each of the terms in Equ. (18) written in a way that is clearly invariant to the unitary transformation used to go from $\varphi_i(\varepsilon)$ to $\psi_i(\varepsilon)$ since,

$$\sum_{i=1}^{l} \sigma_i(\varepsilon_l) = \frac{1}{2i} \ln \det \mathcal{S}(\varepsilon_l) - \frac{1}{2i} \text{Im trace } \mathcal{S}(\varepsilon_l) .$$

If further the elastic scattering phase shifts for zero momentum in each channel are taken to be zero, along with the assumption already made about the energy dependence at various thresholds, equation (18) becomes,

$$\pi(n + N - n_0 - N_0) = \frac{i}{2} [\ln \det \mathcal{S}(\alpha) - \text{Im trace } \mathcal{S}(\infty)] . \tag{19}$$

In Equ. (19) all quantities refer to fixed angular momentum. n, n_0 are the numbers of bound states of H and H_0 respectively. N, N_0 are the numbers of stable states of H which are not stable for H_0 alone and vice-versa. $S(\infty)$ is the S-matrix in any representation for $\varepsilon = \infty$.

* * *

I would like to thank Prof. S. B. TREIMAN for suggesting the possibility of generalizing the Levinson-Jauch theorem at a Theoretical Physics Seminar at the University of Wisconsin and Prof. SACHS for valuable discussions.

RIASSUNTO (*)

Si dimostra che, per un sistema con un numero finito di canali, gli stati legati dell'hamiltoniano libero e completo sono connessi in maniera semplice al determinante ed alla traccia della matrice S.

(*) *Traduzione a cura della Redazione.*

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A Generally Relativistic Field Theory.

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Summary. — In an effort to make contact between field theory and general relativity it is suggested to study space-times embeddable in flat spaces of higher dimensions, as intermediate steps between special and general relativity. The Lorentz invariant action principle formulation of single particle field theory is found to possess a very natural and simple extension to space-times embeddable in a flat six-space. Viewing space-time as a 4-surface in six-space, the metric properties of the former are determined by requiring the action to be stationary with respect to variations of the 4-surface. A more definite theory is obtained by imposing some rather severe restrictions on the form of the Lagrangian. A simple example is studied in detail and is found to describe a pair of Dirac fields interacting through the gravitational field. The main virtue of the theory is that it takes into account the mutual interaction between matter and metric in a consistent manner. It seems plausible that Mach's principle is contained. Second quantization is not carried out, but may be fairly simple by Schwinger's method.

1. – Introduction.

In trying to combine in some way quantum theory with the general theory of relativity it is natural, as with any difficult problem, to do so via several intermediary steps. The stage which may be said to have been reached at present, is the yielding of quantum theory to the requirements of special relativity. It is a very pressing and non-trivial problem to formulate the next stage in such a way that it may be treated more simply than the complete generalization.

A general theorem (1) states that every 4-dimensional Riemannian space V_4 may be immersed in a 10-dimensional pseudo-Euclidean space S_{10}. On the other hand the special theory of relativity may be said to be the result of specializing to such 4-dimensional spaces as may be immersed in a 4-dimensional flat space. This suggests the successive studies of V_4's in S_1, S_5, S_6, S_7, S_8, S_9 and S_{10}, of which the first has been completed and the last is the ultimate aim.

There is a good reason not to expect the study of V_4's imbeddable in an S_5 to be fruitful. It may be proved that if such a V_4 satisfies Einstein's equations, then it is a flat space (2). In contrast, at least three facts indicate that the case of V_4's immersed in an S_6 is sufficiently general to be interesting. First, the Schwartzschild line element, on which the three « crucial » tests of general relativity are based, is a solution of Einstein's equations which is embeddable in an S_6 (3). The cosmological solution of Friedman is also embeddable in S_6 (4). Second, a partial generalization of special relativity which has been much discussed is the extension of the class of equivalent observers to include those with relative uniformly accelerated motion (5). This requires

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(1) It was conjectured by SCHLÄFLI: *Ann. Math.*, (2) **5**, 170 (1871), and proved by CARTAN: *Ann. Soc. Polon. Math.*, **6**, 1 (1927), that a V_n is embeddable locally in an S_m, where $m=\frac{1}{2}n(n+1)$; see *e.g.* L. P. EISENHART: *Riemannian Geometry* (Princeton, 1926), p. 188.

When we say that a space is embeddable in a higher dimensional space we mean that it may be embedded locally. Singular points of the embedding may occur, as well as singular lines, etc. The only restriction is that the manifold of singular points must be of dimension less than that of the embedded space. This is the terminology of C. B. ALLENDOERFER: *Duke Math. Journ.*, **3**, 317 (1937).

(2) E. KASNER: *Am. Journ. Math.*, **43**, 126 (1921).

(3) E. KASNER: *Am. Journ. Math.*, **43**, 130 (1921).

(4) The common cosmological solutions are all embeddable in an S_6, except for the Goedel solution. (Private communication from R. J. FINKELSTEIN).

Having been unable to find a proof in the literature, we give a simple proof that the line-element

$$ds^2 = (dx_1^2 + dx_2^2 + dx_3^2)\psi^2(r, t) - dt^2\varphi^2(r, t) ,$$

is embeddable in an S_6. In fact, take $Z_i = \psi x_i$, to obtain

$$ds^2 = dZ_1^2 + dZ_2^2 + dZ_3^2 + \left[\left(\frac{\partial r\psi}{\partial r}\right)^2 - \psi^2\right] dr^2 + 2\frac{\partial r\psi}{\partial r}\frac{\partial r\psi}{\partial t}\, dr\, dt + \left[\left(\frac{\partial r\psi}{\partial t}\right)^2 - \varphi^2\right] dt^2 .$$

Thus the problem has been reduced to that of embedding a V_2 in an S_3, which is always solvable (ref. (1)).

(5) See for example E. L. HILL: *Phys. Rev.*, **72**, 143 (1947); and J. A. SCHOUTEN: *Rev. Mod. Phys.*, **21**, 421 (1949).

physical laws to be invariant under the group C_{4+} of conformal transformations. This group is isomorphic to the group R_6 of rotations in an S_6 (6). Thirdly, every V_4 which is conformal to a flat space is embeddable in an S_6 (7). It must be stressed that the examples just quoted are only very special cases of V_4's embeddable in and S_6 (8).

The group R_6 of rotations in an S_6 has been found to make contact with theoretical physics in several ways. Perhaps the most important application is due to the close connection with C_{4+}, as was mentioned above. Another connection is found in the circumstance that the simplest two-valued representation of R_6 is on 8-spinors. It has been conjectured that these 8-spinors may correspond to the nucleon wave functions (9). Recently it was pointed out that the group of general co-ordinate transformations can be mapped on the group of special linear homogeneous transformations of the « normal » co-ordinates (10). This group is also isomorphic to R_6.

Whatever the justification for and merits of theories based on invariance under R_6, their success has been checked by the very great difficulty of interpretation. Most attempts have been based on the idea that the space-time world must be a surface of such a kind that the transformations induced on it by R_6 be the group C_{4+}. The transformations which correspond to accelerations are then tentatively associated with some intrinsic property of elementary particles (11), *e.g.* with isotopic spin (9). There are several objections to such identifications even within the philosophy on which they are con-

(6) See for example Y. MURAI: *Progr. Theor. Phys.*, **9**, 147 (1953). The precise statement is that C_{4+} is the covering group of R_6, provided the respective signatures are $+++-$ and $+++-+-$.

(7) L. P. EISENHART: loc. cit., p. 214.

(8) Suppose that a theory can be constructed which will allow us to understand the next stage (that which limits the V_4's to embeddability in an S_6) as well as we actually understand the physics of special relativity. How would one demonstrate the need for a more complete theory? One would have to devise an experiment to explore a phenomenon which is inconsistent with the metric being embeddable in an S_6. Such an experiment would have to be quite difficult. At present it is probably not even possibe to think of such an experiment. This is because we are unable to formulate a physical problem in a space in which a cosmic time does not exist, which is the reason why Friedman's cosmological solution is embeddable in an S_6. We might therefore conjecture that space-time is not a general V_4, but a V_4 which may be thought of as a continuous sequence of 3-spaces. This has the consequence that space-time is embeddable in an S_6, and excludes *e.g.* the Goedel solution. This conjecture not only is in very little danger of being contradicted, but has the very attractive feature of restricting the general theory of relativity just enough that we may understand it. This conjecture, however, is not the premise of the present paper.

(9) See for example R. L. INGRAHAM: *Phys. Rev.*, **106**, 595 (1957).

(10) R. J. FINKELSTEIN: *Phys. Rev.*, **109**, 1842 (1958).

(11) See for example Y. MURAI: *Progr. Theor. Phys.*, **11**, 441 (1954).

ceived ([12]), although these might conceivably be overcome by a particularly ingenious interpretation. Here, however, we only note that any *fixed* correspondence between quantities in S_6 and observables in V_4 is inconsistent with our present aims. The most important feature of general relativity which we hope to incorporate in a future theory is that the metric is calculable and depends on the matter distribution ([13]). Hence the form of the 4-dimensional surfaces in S_6 is to be calculated within the theory, and not axiomatically fixed *a priori*.

2. – A generalized action principle.

Conventional field theory is based on the following considerations. Let a partial specification of a physical system on a 3-space σ be represented by the symbol $\Psi(\sigma)$. The objects under study are the correlations that exist between, say, $\Psi(\sigma_1)$ and $\Psi(\sigma_2)$. Such correlations are represented by the symbol $U(\sigma_2, \sigma_1)$. It is conjectured that to these symbols correspond well-defined mathematical quantities, so that

$$\Psi(\sigma_2) = U(\sigma_2, \sigma_1)\Psi(\sigma_1) .$$

The transformation function $U(\sigma_2, \sigma_1)$ may be generated from an infinitesimal transformation

$$\delta\Psi(\sigma) = \delta U(\sigma, \sigma_0)\Psi(\sigma_0) ,$$

where σ_0 serves as a reference.

The σ's are restricted to be embeddable in an S_4, and may be labelled by a single parameter t. Every space-like surface may be considered as a surface in S_4 determined by $t = t\,(x, y, z)$, where t, x, y, z are the co-ordinates in S_4. As different surfaces are considered in succession, one may speak of a development of σ in t. It is easy to see that $\delta U(\sigma_2, \sigma_1)$ must have the form of a volume integral

$$\delta U(\sigma_2, \sigma_1) = \delta \int_{\sigma_1}^{\sigma_2} L \, \mathrm{d}^4x , \tag{2.1}$$

in S_4. Since the co-ordinates in S_4 are determined by the embedded σ only

([12]) See for example R. Omnès: *Compt. Rend.*, **245**, 1382 (1957).

([13]) At least to a certain extent, although one would hope to be able to incorporate Mach's principle.

up to a rigid displacement in S_4 [14], this expression makes sense only if $L\,\mathrm{d}^4x$ is invariant. If d^4x is the invariant volume element, L must be an invariant.

In the introduction the appropriateness of the S_6-approximation to general relativity was argued. It is expected that a field theory in such spaces will retain many of the mathematical features of ordinary field theory. One should therefore select a point in the development of the conventional theory where the flatness of 4-space is explicitly assumed, and attempt a generalization. The success of such an attempt may depend critically on the point selected; it is probably important that the generalization be as direct and as obvious as possible. The scope of the present work is merely to investigate one such point, namely that at which the space-like σ-spaces are assumed to be embeddable in an S_4. This point would seem particularly appropriate to the S_6-approximation, since an S_6 is just what is needed to embed an arbitrary 3-dimensional Riemannian space.

Having thus extended the class of allowed space-like σ-spaces, we have the new situation that not all σ-spaces are surfaces in space time. Instead space time is one of many possible « path »'s in the manifold of σ-spaces, and a new dynamical principle is needed to select the actual « path ». Fortunately, the action principle discussed above supplies this automatically. To see this note that the integral (2.1) can be viewed geometrically as a surface integral over a four-dimensional surface (*i.e.* space-time) in 6-space. In keeping with the general method of considering all possible variations in (2.1), one must carry out, in addition to the conventional variations, the variation of the « path » of integration. Thus the form of the 4-dimensional surface, *i.e.* the metric of space-time, is determined by making the action principle stationary with respect to the new variation.

Up to now we have only a very general framework for a theory. A definite physical theory is obtained only after restricting in some way the form of the Lagrangian density. The remainder of this paper is devoted to the study of a particular theory, obtained by a set of restrictions imposed on the form of L.

In general L is a function of a certain number of field variables, the components of the metric tensor, and the space-time co-ordinates. As the first restriction we write

(I) $$L = L(\varphi, \varphi_i; g_{ij}, x^i)\,.$$

Here x_i and g_{ik} are the space-time co-ordinates and metric tensor; φ stands for the whole set of field variables, and φ_i for $\partial\varphi/\partial x^i$. The meaning of (I) is

[14] It is even less determined if the metric tensor on σ is degenerate.

that L may involve only φ, the first derivatives of φ, plus g_{ik} and x_i in an arbitrary way. This restriction is always made in conventional theory, and is due mainly to the fact that a more general variational principle is not manageable or well understood.

Instead of (2.1) one may always write

$$\delta U(\sigma_2, \sigma_1) = \delta \int_{\sigma_1}^{\sigma_2} \frac{1}{2} L^{\alpha\beta} \mathrm{d}\sigma_{\alpha\beta} , \tag{2.2}$$

where $\mathrm{d}\sigma_{\alpha\beta}$ is the 4-dimensional surface element, and $L^{\alpha\beta}$ is a skew 6-dimensional tensor. (Latin indices run from 1 to 4, Greek indices from 1 to 6). We now impose the crucial second restriction:

(II) The Lagrangian tensor in (2.2) is independent of the path .

This does not mean that the Lagrangian *density* is independent of the properties of space-time, but it does specify this dependence completely, once $L^{\alpha\beta}$ is known. The meaning of (II) is that no inherent properties, such as a « surface tension », are ascribed to the space-time surface. The inclusion of a free gravitational part is also excluded. It is, of course, nevertheless to be hoped that the conditions on the space-time metric, obtained by varying the path of integration in (2.2), be similar to those of general relativity. The main virtue of the present theory, we feel, is that the mutual interaction of matter and metric is treated in a self consistent manner.

Writing z^ν for the 6-space co-ordinates, let space-time be the surface

$$z^\nu = Z^\nu(x) . \tag{2.3}$$

where $Z^\nu(x)$ are six reasonably continuous functions of the four parameters x^i. With $Z^\nu{}_i \equiv \partial Z^\nu / \partial x^i$, the 4-dimensional surface element is

$$\begin{cases} \mathrm{d}\sigma_{\alpha\beta} = \varepsilon_{\alpha\beta} \mathrm{d}^4 x , \\ \varepsilon^{\alpha\beta} = \varepsilon_{\alpha\beta;\delta\varepsilon\eta} Z^\gamma{}_1 Z^\delta{}_2 Z^\varepsilon{}_3 Z^\eta{}_4 . \end{cases} \tag{2.4}$$

Similarly for the 3-dimensional surface element

$$\begin{cases} \mathrm{d}\sigma_{\alpha\beta\gamma} = \varepsilon_{\alpha\beta\gamma}{}^i \mathrm{d}\sigma_i , \\ \varepsilon_{\alpha\beta\gamma}{}^i = \frac{1}{3!} \varepsilon_{\alpha\beta\gamma\delta\varepsilon\eta} \varepsilon^{ijkl} Z^\delta{}_j Z^\varepsilon{}_k Z^\eta{}_l . \end{cases} \tag{2.5}$$

Here d^4x is the invariant volume element of space-time, and $\mathrm{d}\sigma_i$ is the 4-vector surface element of a 3-dimensional surface in space-time. The following formulae are useful

$$\text{(2.6)}\qquad \begin{cases} g_{ij} &= g_{\mu\nu} Z^{\mu}{}_{i} Z^{\nu}{}_{j}\,, \\ g &= \tfrac{1}{2}\varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta}\,, \\ g^{ij} &= \left(\frac{1}{3!\,g}\right)\varepsilon_{\alpha\beta\gamma}{}^{i}\varepsilon^{\alpha\beta\gamma j}\,, \\ [ij,\,k] &= g_{\mu\nu} Z^{\mu}{}_{ij}\, Z^{\nu}{}_{k}\,, \\ Z^{\nu}{}_{i,j} &= {}^{2}\delta_{\mu}{}^{\nu} Z^{\mu}{}_{ij}\,. \end{cases}$$

In the last formula appears the normal projection of the Kronecker symbol. The latter may be split into a normal and a tangential part as follows

$$\text{(2.7)}\qquad \delta_{\mu}{}^{\nu} = {}^{1}\delta_{\mu}{}^{\nu} + {}^{2}\delta_{\mu}{}^{\nu} = Z^{\nu}{}_{i} Z_{\mu}{}^{i} + \frac{1}{g}\varepsilon_{\mu\alpha}\varepsilon^{\nu\alpha}\,.$$

Covariant differentiation with respect to x^i is denoted by a subscript following a comma. The product of $\mathrm{Det}\, g_{\mu\nu}$ and $\mathrm{Det}\, g_{ij}$ is denoted g.

Varying both the field quantities φ and the surface functions (2.3), calculation of the r.h.s. of (2.2) gives the result (according to I and II, $L^{\alpha\beta}$ depends on $\varphi(z)$ and on $\varphi_{\nu}(z) \equiv \partial\varphi(z)/\partial z^{\nu}$ only):

$$\text{(2.8)}\qquad \delta U(\sigma_2,\,\sigma_1) = \int_{\sigma_1}^{\sigma_2} \Big\{ \left(\frac{\partial L}{\partial\varphi} - \Pi^{i}{}_{,i}\right)\delta\varphi + (\Pi^{i}\delta\varphi)_{,i} + \\ + \frac{\partial L}{\partial\varphi_{\gamma}}\,{}^{2}\delta_{\gamma}{}^{\mu}\,\frac{\partial\,\delta\varphi}{\partial z^{\mu}} + \varepsilon_{\nu\gamma}\,\frac{\partial L^{\alpha\beta}}{\partial z^{\beta}}\,\delta Z^{\gamma} + \frac{1}{2}\,(L^{\alpha\beta}\varepsilon_{\nu\beta\gamma}{}^{i}\,\delta Z^{\gamma})_{,i} \Big\}\,\mathrm{d}^4x\,.$$

Here L stands for $\frac{1}{2}\varepsilon_{\alpha\beta}L^{\alpha\beta}$. The momenta are defined by

$$\text{(2.9)}\qquad \Pi^{i} \equiv \frac{\partial L}{\partial\varphi_{\gamma}}\,Z_{\gamma}{}^{i}\,.$$

Because L is a scalar density, the covariant divergences occuring in the integral actually reduce to ordinary divergences, so that surface terms may be identified. The meaning of ${}^{2}\delta_{\gamma}{}^{\mu}\,\partial\delta\varphi/\partial z^{\mu}$ is the gradient of $\delta\varphi$ normal to the 4-surface, which is evidently independent of $\delta\varphi$ on that surface. Hence, taking variations which vanish on σ_1 and on σ_2 we get a field equation, a subsidiary condition and a path equation:

Field Equation:

$$\text{(2.10)}\qquad \frac{\partial L}{\partial\varphi} - \Pi^{i}{}_{,i} = 0\,,$$

Subsidiary Condition:

$$(2.11) \qquad {}^{2}\delta_{\gamma}{}^{\mu}\frac{\partial L}{\partial \varphi_{\gamma}} = 0\,,$$

Path Equation:

$$(2.12) \qquad \varepsilon_{\alpha\gamma}\frac{\partial L^{\alpha\beta}}{\partial z^{\beta}} = 0\,.$$

If (2.12) is ignored, and a flat space is assumed in (2.10) and (2.11), the usual flat space theory is formally obtained. Of course, there is no assurance that there exists an $L^{\alpha\beta}$ corresponding to every L in the conventional theory. It is, in fact, very probable that only a very restricted class of L's may be derived from $L^{\alpha\beta}$'s. This is because of the very strong restriction II, which may eventually turn out to be too severe. Until that has been clearly shown, however, we think that it may deserve further study. In the next section detailed study of the main equations (2.10), (2.11) and (2.12) shall lead us to a third, and last, and much less severe restriction on the structure of $L^{\alpha\beta}$.

3. – The structure of $L^{\alpha\beta}$.

It is well known that the Lagrangian density in ordinary field theory can always, by an appropriate choice of field variables, be written in the form

$$(3.1) \qquad L = i\tilde{\varphi}A^{\mu}p_{\mu}\varphi - \mathcal{H}(\varphi, \tilde{\varphi})\,,$$

where $\mathcal{H}$ contains φ and $\tilde{\varphi}$ but not their derivatives. Similarly the Lagrangian tensor $L^{\alpha\beta}$ can always be written

$$(3.2) \qquad L^{\alpha\beta} = i\tilde{\varphi}A^{\alpha\beta\mu}p_{\mu}\varphi - \mathcal{H}^{\alpha\beta}(\varphi, \tilde{\varphi})\,,$$

where

$$(3.3) \qquad \tilde{\varphi} \equiv \varphi^{\dagger}\eta\,,$$

and η is a numerical matrix to be determined in such a way that L be hermitian. The numerical matrix $A^{\alpha\beta\mu}$ and the «Hamiltonian» $\mathcal{H}^{\alpha\beta}$ are both skew in α, β. The Lagrangian density

$$L - \tfrac{1}{2}\varepsilon_{\alpha\beta}L^{\alpha\beta}$$

is given by (3.1), if we define

$$(3.4) \qquad A^{\mu} \equiv \tfrac{1}{2}\varepsilon_{\alpha\beta}A^{\alpha\beta\mu}, \qquad \mathcal{H} \equiv \tfrac{1}{2}\varepsilon_{\alpha\beta}\mathcal{H}^{\alpha\beta}.$$

The subsidiary condition

$$\varepsilon_{\alpha\gamma}(\partial L/\partial\varphi_\gamma) = 0$$

becomes

$$\varepsilon_{\mu\gamma}\varepsilon_{\alpha\beta}A^{\alpha\beta\mu}\varphi = 0\,. \tag{3.5}$$

Although it may not be absolutely necessary, we shall limit ourselves to Lagrangians such that (2.5) is identically satisfied:

$$\varepsilon_{\mu\gamma}\varepsilon_{\alpha\beta}A^{\alpha\beta\mu} = 0\,,$$

Since $A^{\alpha\beta\mu}$ is independent of co-ordinates the factor $\varepsilon_{\mu\nu}\varepsilon_{\alpha\beta}$ may be deleted, provided due regard is given to the symmetries of this factor. We obtain

Restriction III:

$$A^{\alpha\beta\mu} = A^{[\alpha\beta\mu]}\,, \tag{III}$$

where the right hand side is, by definition, completely antisymmetric. In the following paragraphs this restriction to completely antisymmetric $A^{\alpha\beta\mu}$ will be seen to produce a significant simplification. Indeed it is probable that (III) is a necessary condition for the path equation to be meaningful.

The field equation corresponding to (3.2) is

$$\frac{1}{2}\varepsilon_{\alpha\beta}A^{\alpha\beta\mu}\varphi_\mu - \frac{1}{4}A^{\alpha\beta\mu}(\varepsilon_{\alpha\beta}Z_\mu{}^i)_{,i}\varphi - \frac{\partial\mathcal{H}}{\partial\tilde{\varphi}} = 0\,. \tag{3.6}$$

The second term vanishes by virtue of the antisymmetric of $A^{\alpha\beta\mu}$. The first term involves tangential derivatives only. Writing

$$A^i = Z_\alpha{}^i A^\alpha\,,$$

the field equation (6) reduces to

$$A^i\varphi_i = \frac{\partial\mathcal{H}}{\partial\tilde{\varphi}}\,, \tag{3.7a}$$

$$\tilde{\varphi}_i A^i = -\frac{\partial\mathcal{H}}{\partial\varphi}\,. \tag{3.7b}$$

Turning next to the path equation (2.7), we separate the terms which formally involve normal derivations, namely

$$\varepsilon_{\alpha\gamma}{}^2\delta_\beta{}^\delta\frac{\partial L^{\alpha\beta}}{\partial z_\delta} + \varepsilon_{\alpha\gamma}\frac{\partial}{\partial z^\beta}(i\tilde{\varphi}A^{\alpha\beta\mu}p_\varrho\varphi),$$

because of the antisymmetry of $A^{\alpha\beta\mu}$ this expression contains no second order derivatives, and reduces to

$$\frac{1}{2}\varepsilon_{\alpha\gamma}{}^{2}\delta_{\beta}{}^{\delta}(\tilde{\varphi}_{\delta}A^{\alpha\beta\mu}\varphi_{\mu}-\tilde{\varphi}_{\mu}A^{\alpha\beta\mu}\varphi_{\delta})-\varepsilon_{\alpha\gamma}{}^{2}\delta_{\beta}{}^{\delta}\left(\frac{\partial\mathcal{H}^{\alpha\beta}}{\partial\varphi}\varphi_{\delta}+\tilde{\varphi}_{\delta}\frac{\partial\mathcal{H}^{\alpha\beta}}{\partial\tilde{\varphi}}\right)+$$

$$+\frac{1}{2}\varepsilon_{\alpha\gamma}{}^{2}\delta_{\beta}{}^{\delta}(\tilde{\varphi}_{\beta}A^{\alpha\beta\mu}\varphi_{\delta}-\tilde{\varphi}_{\delta}A^{\alpha\beta\mu}\varphi_{\beta}).$$

The first and the last terms are equal by the antisymmetry of $A^{\alpha\beta\mu}$, and we obtain

$$\varepsilon_{\alpha\gamma}{}^{2}\delta_{\beta}{}^{\delta}\left\{\tilde{\varphi}_{\delta}\left(A^{\alpha\beta\mu}\varphi_{\mu}-\frac{\partial\mathcal{H}^{\alpha\beta}}{\partial\tilde{\varphi}}\right)-\left(\tilde{\varphi}_{\mu}A^{\alpha\beta\mu}+\frac{\partial\mathcal{H}^{\alpha\beta}}{\partial\varphi}\right)\varphi_{\delta}\right\}.$$

Use of the identity

$$\varepsilon_{\alpha\gamma}{}^{2}\delta_{\beta}{}^{\delta}-\varepsilon_{\beta\gamma}{}^{2}\delta_{\alpha}{}^{\delta}+\varepsilon_{\beta\alpha}{}^{2}\delta_{\gamma}{}^{\delta}=0\,, \tag{3.8}$$

enables us to permute the indices γ, β in the first factor, upon which the whole expression is seen to vanish as a consequence of (2.7). The remainder of the path equation, which does not involve normal derivatives, reads

$$\varepsilon_{\alpha\gamma}\left\{\tilde{\varphi}_{i}A^{\alpha ij}\varphi_{j}-Z_{\beta}{}^{i}\frac{\partial\mathcal{H}^{\alpha\beta}}{\partial x^{i}}\right\}=0\,. \tag{3.9}$$

Summarizing, the effect of (III) is *a*), to reduce the subsidiary condition to an identity and *b*), to ensure the absence of normal derivatives from the Lagrangian density (and hence from the field equation) as well as from the path equation.

A special class of Lagrangians may be obtained in the following way. Every first order « wave-equation » in 6-space:

$$(\Gamma^{\mu}p_{\mu}+im)\varphi=0 \tag{3.10}$$

defines a set of six form invariant numerical matrices. From these may be formed

$$A^{\alpha\beta\mu}=\Gamma^{[\alpha}\Gamma^{\beta}\Gamma^{\mu]}=\frac{1}{3!}\sum\Gamma^{\alpha}\Gamma^{\beta}\Gamma^{\mu}, \tag{3.11}$$

where the sum is the alternating sum over all permutations of α, β, μ.

The remainder of this paper is concerned with a particular example; except for the digression of the next section, the most useful result of which is a general definition of the covariant derivative.

4. – Digression on motions.

Although the main purpose of this section is to give a general definition of the covariant derivative of field quantities, for use in subsequent sections, this question is discussed in the slightly more general context of arbitrary local motions. Consider a 4-dimensional space-time embedded in an S_6, and a 3-dimensional space-like surface σ in space-time. The set of all points with co-ordinates

$$x'^i = x^i + \delta x^i(x) , \tag{4.1}$$

where x^i is a point on σ and $\delta x^i(x)$ is an infinitesimal-valued function, defines a new surface σ'. Alternatively (4.1) may be viewed as a co-ordinate transformation. Adopting an intermediary interpretation of (4.1) we fix our attention on a particular point $x^i = {}^0x^i$ on σ, and say that (4.1) « moves » the point ${}^0x^i$ to the point ${}^0x'^i = {}^0x^i + \delta x^i({}^0x)$, and in addition defines a co-ordinate transformation in an infinitesimal neighbourhood of ${}^0x^i$. This co-ordinate transformation is given by the derivatives of the function $\delta x^i(x)$, evaluated at ${}^0x^i$. It is convenient to choose the co-ordinate transformation in such a way that certain mathematical quantities have the same values at ${}^0x'$ in the new coordinate system as they had at ${}^0x$ in the old. Thus it is a well known result that

$$\delta g_{ij}(x)\,|_{x={}^0x} \equiv g'_{ij}({}^0x') - g_{ij}({}^0x) = 0 \tag{4.2}$$

if and only if Killing's equations are satisfied at ${}^0x$:

$$\delta x_{i,j} + \delta x_{j,i}\,|_{x={}^0x} = 0 . \tag{4.3}$$

A simple calculation shows that

$$\delta \Gamma_{ij}{}^k(x)\,|_{x={}^0x} \equiv \Gamma'_{ij}{}^k({}^0x') - \Gamma_{ij}{}^k({}^0x) = 0 \tag{4.4}$$

if and only if

$$\delta x_{i,jk} + \delta x_{k,ij} - \delta x_{k,ji}\,|_{x={}^0x} = 0 . \tag{4.5}$$

Invariance, in the sense of (4.2) and (4.4), may be extended to mixed variables, *i.e.* to variables with both 4-vector and 6-vector indices. The 4-surface is determined by the equations

$$z^\alpha = Z^\alpha(x) . \tag{4.6}$$

If a particular point ${}^0x$ on σ is characterized by the 6-co-ordinates ${}^0z^\nu$, the new point ${}^0x'$ on σ' will have the different co-ordinates

$$ {}^0z'^\alpha = {}^0z^\alpha + Z^\alpha{}_i({}^0x)\delta x^i({}^0x) . \tag{4.7} $$

Introduce a rigid transformation of the 4-surface to compensate for this difference:

$$ z^\alpha = y^\alpha + b^\alpha + b^\alpha{}_\beta(y_\alpha - {}^0y^\alpha) , \tag{4.8} $$

where b^α and $b^\alpha{}_\beta$ are infinitesimal and constant, and

$$ b^{\alpha\beta} = - b^{\beta\alpha} . \tag{4.9} $$

We obtain

$$ {}^0y'^\alpha = {}^0z^\alpha , $$

by taking

$$ b^\alpha = Z^\alpha{}_i \delta x^i |_{x={}^0x} . \tag{4.10} $$

while the $b^\alpha{}_\beta$ remain arbitrary. The equation for the 4-surface is

$$ y^\alpha = Z^\alpha(x) - b^\alpha - b^\alpha{}_\beta[Z^\beta(x) - Z^\beta({}^0x)] \equiv Y^\alpha(x) . \tag{4.11} $$

Hence

$$ \delta Z^\alpha(x)|_{x={}^0x} \equiv Y^\alpha({}^0x') - Z^\alpha({}^0x) = 0 \tag{4.12} $$

which expresses the invariance of $Z^\alpha(x)$. If Killing's equations (4.3) are satisfied, it may be arranged that

$$ \delta Z^\alpha{}_i(x)|_{x={}^0x} = 0 , \tag{4.13} $$

by taking for $b^\alpha{}_\beta$ the following expression:

$$ b^\alpha{}_\beta = \{ Z^\alpha{}_j Z_\beta{}^i \, \delta x^j{}_{,i} + (Z^\alpha{}_{i,j} Z_\beta{}^i - Z^\alpha{}_i Z_\beta{}^i{}_{,j}) \, \delta x^j \}|_{x={}^0x} + c^\alpha{}_\beta . \tag{4.14} $$

Here the first term represents a rotation in a plane tangential to the « path » at ${}^0x$. The second and third terms represent rotations in planes with one tangential and one normal direction, respectively two normal directions. The magnitude of the latter remains arbitrary. Finally we investigate the variation of $Z^\alpha{}_{ij}(x)$. The invariance of the tangential projection:

$$ \delta({}^1\delta_\beta{}^\alpha Z^\beta{}_{ij})|_{x={}^0x} = 0 \tag{15} $$

is a consequence of (4) and (13). The variation of the normal projection

$$^2\delta_\beta{}^\wedge Z^\beta{}_{ij} = Z^\beta{}_{i,j}$$

does not vanish, however. Taking $c^\alpha{}_\beta = 0$, we find

$$\delta Z^\alpha{}_{i,j}(x)\,|_{x={}^0x} = \{{}^2\delta_\beta{}^\alpha Z^\beta{}_{i,jk}\delta x^k + Z^\alpha{}_{i,k}\,\delta x^k{}_{,j} + Z^\alpha{}_{j,k}\,\delta x^k{}_{,i}\}\,|_{x={}^0x}\,. \tag{4.16}$$

Hence $Z^\wedge{}_{i,j}(x)$ is a genuine dynamical variable, as is made plausible by noting the following expression for the curvature tensor

$$R_{ijkl} = g_{\alpha\beta}(Z^\wedge{}_{i,k}\,Z^\beta{}_{j,l} - Z^\alpha{}_{i,l}\,Z^\beta{}_{j,k})\,. \tag{4.17}$$

By a local motion near the point $x = {}^0x$ we shall mean the transformation of a space-like surface σ passing through ${}^0x$ into the space-like surface σ' defined by (1), where $\delta x^i(x)$ differs from zero only infinitesimally close to ${}^0x$ and satisfies (3) and (5); followed by a rigid displacement of the entire 4-surface given by (8), (10) and (14). A local translation is characterized by

$$\delta x_{i,j}\,|_{x={}^0x} = 0\,, \qquad \text{(for translations),} \tag{4.18}$$

and a local rotation by

$$\delta x^i\,|_{x={}^0x} = 0, \qquad \text{(for rotations).} \tag{4.19}$$

Consider an arbitrary geometrical object $\varphi(z)$ in six-space, and let $\varphi(x)$ denote its values on the space-time four-surface. If $s^\beta{}_\alpha$ stands for the 6-spin matrices associated with $\varphi(z)$, and $s^i{}_j$ for their space-time projections, the total change of $\varphi(x)$ under a local motion is

$$\delta\varphi(x)\,|_{x={}^0x} = \{-\delta x^l\nabla_l\varphi(x) + \tfrac{1}{2}\,\delta x^j{}_{,i}\nabla^i{}_j\varphi(x)\}\,|_{x={}^0x}\,, \tag{4.20}$$

where

$$\nabla_l\varphi(x) = \frac{\partial}{\partial x^l}\,\varphi(x) - \frac{1}{i}\,Z^\alpha{}_{i,l}\,Z_\beta{}^i s^\beta{}_\alpha\varphi(x)\,, \tag{4.21}$$

$$\nabla_{ij}\varphi(x) = [(x_i - {}^0x_i)\nabla_j - (x_j - {}^0x_j)\nabla_i]\varphi(x) - \frac{1}{i}\,s_{ij}\varphi(x)\,. \tag{4.22}$$

The operator ∇_l is the operator of completely covariant differentiation. It contains the conventional definition of the covariant derivative of tensors in the following way, *e.g.*

$$\nabla_l\varphi^i \equiv \nabla_l(Z_\alpha{}^i\varphi^\alpha) = Z_\alpha{}^i\nabla_l\,\varphi^\alpha = \varphi^i{}_{,l}\,. \tag{4.23}$$

For a geometrical object with indices referring to 4-space, to 6-space or to both we have finally

$$\nabla_l \varphi(x) = \varphi(x)_{,l} - \frac{1}{i} Z^{\alpha}{}_{i,l} Z_{\beta}{}^{i} s^{\beta}{}_{\alpha} \varphi(x). \tag{4.24}$$

The completely covariant derivatives of g_{ij}, $[ij, k]$, $Z^{\alpha}(x)$ and $Z^{\alpha}{}_{i}$ are all zero, while

$$\nabla_l Z^{\alpha}{}_{i,j} = {}^{2}\delta_{\beta}{}^{\alpha} Z^{\beta}{}_{i,jl} . \tag{4.25}$$

It may be verified that

$$\nabla_l s_{ij} = 0 . \tag{4.26}$$

The set of local motions referred to a point ${}^{0}x$ almost form a group in the sense that the commutators, *e.g.*

$$[\nabla_i, \nabla_j]\varphi(x) ,$$

evaluated at $x = {}^{0}x$, be linearly expressible in terms of ∇_i and ∇_{ij}, evaluated at $x = {}^{0}x$. A group is obtained by adjunction of a single operator

$$\xi = i(-g)^{\frac{1}{2}} \varepsilon_{\alpha\beta} s^{\alpha\beta} . \tag{4.27}$$

The commutation relations are

$$[\nabla_i, \nabla_j]\varphi(x)|_{x={}^{0}x} = \{\tfrac{1}{2} R_{ijkl} \nabla_{kl} + \varkappa_{ij} \xi\}\varphi(x)|_{x={}^{0}x} , \tag{28a}$$

$$[\nabla_i, \nabla_{jk}]\varphi(x)|_{x={}^{0}x} = \{g_{ij} \nabla_k - g_{ik} \nabla_j\}\varphi(x)|_{x={}^{0}x} , \tag{28b}$$

$$[\nabla_{ij}, \nabla_{kl}]\varphi(x)|_{x={}^{0}x} = \{g_{jk}\nabla_{il} + g_{il}\nabla_{jk} - g_{ik}\nabla_{jl} - g_{jl}\nabla_{ik}\}\varphi(x)|_{x={}^{0}x} , \tag{4.28c}$$

$$[\nabla_i, \xi]\varphi(x) = [\nabla_i, \xi]\varphi(x) = 0 , \tag{4.28d}$$

where $\varkappa_{ij}$ are the functions

$$\varkappa_{ij} = \tfrac{1}{2}(-g)^{-\frac{1}{2}} \varepsilon_{\alpha\beta} Z^{\alpha}{}_{k,j} Z^{\beta}{}_{l,i} g^{kl} . \tag{4.29}$$

5. – Free 8-spinor field.

As a first example let (3.10) describe a spinor field. Then the Γ^{μ} are 8-by-8 anticommuting hermitian matrices. Taking

$$\mathcal{H}^{\alpha\beta} = \overline{\psi} \tfrac{1}{2} m(\Gamma^{\alpha}\Gamma^{\beta} - \Gamma^{\beta}\Gamma^{\alpha})\psi , \tag{5.1}$$

or

$$\mathcal{H} = \bar{\psi} m \sqrt{-g}\, \xi \psi\,, \qquad \xi \equiv \frac{1}{2\sqrt{-g}} \varepsilon_{\alpha\beta} \Gamma^{\alpha} \Gamma^{\beta}, \tag{5.2}$$

we obtain, using (3.11), the Lagrangian density

$$L = i\bar{\psi} \sqrt{-g}\, \xi (\Gamma^{i} p_{i} + im) \psi\,. \tag{5.3}$$

A very useful formula is

$$\varepsilon_{\alpha\beta} \varepsilon_{\gamma\delta} = g({}^{2}g_{\alpha\gamma}\, {}^{2}g_{\beta\delta} - {}^{2}g_{\alpha\delta}\, {}^{2}g_{\beta\gamma})\,. \tag{5.4}$$

By means of (4) it becomes trivial to prove that $\xi^{2} = 1$. Furthermore Γ^{i} commute with ξ. The field equation is

$$\left\{ \Gamma^{i} \frac{\partial}{\partial x^{i}} + \frac{1}{2} \xi (\xi \Gamma^{i})_{,i} - m \right\} \psi = 0\,. \tag{5.5}$$

The vanishing of the second term was pointed out already in the general theory. It is nevertheless convenient to study separately the two terms obtained by differentiating ξ and Γ^{i}. Thus we define

$$\Sigma_{i} \equiv -\tfrac{1}{2} \xi \xi_{,i} \tag{5.6}$$

and verify, noting that ξ is a scalar, that

$$\Sigma_{i} = \tfrac{1}{2} \Gamma^{j}{}_{,i} \Gamma_{j}\,. \tag{5.7}$$

Since $\Gamma^{j}{}_{,i}$ anticommutes with Γ_{k}, there follows that

$$[\Sigma_{j}, \Gamma_{i}] = \Gamma_{i,j}\,. \tag{5.8}$$

This property of Σ_{i} allows us to define spinor-covariant differentiation in such a way that the spinor covariant derivative of Γ_{i} vanishes. Retaining the notation « $_{,j}$ » for tensor-covariant differentiation, we introduce an operator ∇_{i} in the following way. If $\psi^{\cdots}$ stands for a spinor-tensor with any number of tensor indices:

$$\nabla_{i} \psi^{\cdots} = (\psi^{\cdots})_{,i} - \Sigma_{i} \psi^{\cdots}\,. \tag{5.9}$$

With this notation (8) becomes

$$[\nabla_i, \Gamma^j] = 0\,, \tag{5.10}$$

and the field equation may be written in two equivalent forms:

$$\left(\Gamma^i \frac{\partial}{\partial x^i} - m\right)\psi = 0\,, \tag{5.11}$$

or

$$(\Gamma^i \nabla_i - m)\psi = \tfrac{1}{2}\Gamma^i{}_{,i}\psi\,. \tag{5.12}$$

We may verify that the present definition of the operator ∇_i agrees with the general definition (4.21). Inserting the expression

$$s^\beta{}_\alpha = \frac{1}{4i}(\Gamma^\beta \Gamma_\alpha - \Gamma_\alpha \Gamma^\beta)\,, \tag{5.13}$$

for the spin operator into (4.21) gives

$$\nabla_i \psi = \frac{\partial}{\partial x^i}\psi - \frac{1}{2}\Gamma^j{}_{,i}\Gamma_j \psi, \tag{5.14}$$

which does indeed coincide with the present definition. The matrices Σ_i correspond to what is conventionally denoted Γ_i.

In order to interpret the theory, we must exhibit the reduction of the wave-function according to the Lorentz-group. Because Ψ furnishes an irreducible representation of the six-dimensional « Lorentz » group this is to diagonalize the projection of (13) on the normal plane. This projection is equal to $\frac{1}{2}\xi$. Hence the irreducible parts of ψ are obtained by the projections:

$$\psi_\pm \equiv \tfrac{1}{2}(1 \pm \xi)\psi\,. \tag{5.15}$$

Verifying that

$$\xi\Gamma^i\nabla_i = \Gamma^i\nabla_i\xi\,, \qquad \xi\Gamma^i{}_{,i} = -\Gamma^i{}_{,i}\xi\,, \tag{5.16}$$

we may write the field equation (12) in four-component form

$$\begin{cases} (\Gamma^i\nabla_i - m)\psi_+ = \tfrac{1}{2}\Gamma^i{}_{,i}\psi_-\,, \\ (\Gamma^i\nabla_i - m)\psi_- = \tfrac{1}{2}\Gamma^i{}_{,i}\psi_+\,. \end{cases} \tag{5.17}$$

Hence the present example describes a pair of Dirac four-spinor fields interacting through the « gravitational field ».

By a fairly straightforward calculation the path equation (2.12) may be reduced to

$$(5.18) \qquad {}_{3\alpha\gamma}[m^2\overline{\psi}\Gamma^\alpha\psi + g^{ij}\overline{\psi}_i\Gamma^\alpha\psi_j] .$$

The simplicity of the set (5.11), (5.18) is remarkable, indicating the possibility of obtaining at least approximate solutions. In a flat space any plane wave solution of (11) is also a solution of (5.18). As soon as one tries to obtain a localized solution by forming a wave packet, however, it becomes impossible to satisfy (5.18). This very reasonable feature of the theory nourishes our hope that Mach's principle may be contained.

It is very simple to construct another theory which may be interpreted as describing a pair of two-component neutrino fields. Equation (3.10) may also be interpreted as a Kemmer equation. The resulting theory has been investigated in detail, but the physical interpretation is not immediately obvious.

6. – Discussion.

The main idea of the theory just presented is that space-times which are locally embeddable in an S_6 are probably sufficiently general that a physical theory in such space-times must exhibit all important features of a generally relativistic theory. If this limited class of space-times can be fully understood, the ultimate generalization is probably a simple matter. What is not the premise of this paper, though an interesting speculation, is the possibility that such final generalization may not be needed. Usually in physics, when the need arises to generalize a theory, it is imprudent to carry the generalization further than absolutely necessary. There is no experimental evidence that space-time is not locally embeddable in an S_6, and the existence of a cosmic time is very reassuring.

In attempting to build a physical theory in the S_6 approximation, the Lorentz invariant action principle discussed in Section 2 suggests a very direct and obvious generalization. Special relativity is viewed as arising from restricting 3-space to embeddability in an S_4. Lifting this restriction an action principle in S_6 is obtained, and with it is automatically included the possibility—or indeed the necessity—of varying the « path » of integration. To each « path » corresponds a metric structure of space-time, and that which makes the action stationary is interpreted as that which is actually realized. It is hard to imagine how a lesser departure from conventional theory can be accomplished.

The further development of the theory is not nearly so obvious. We have chosen to reduce the field of possible theories by imposing certain conditions on the Lagrangian. Excluding higher derivatives of field quantities than the first from L (Restriction I) is necessary to make the theory susceptible to a physical interpretation. Replacement of L by the Lagrangian tensor $L^{\alpha\beta}$ is no restriction. Restriction I is then nearly equivalent to the requirement that $L^{\alpha\beta}$ depend on the field quantities and their first derivatives with respect to the six-co-ordinates only. (This near equivalence becomes complete after Restriction III is imposed). Restriction II is probably as controversial as it is restrictive. It is based on an analogy with classical mechanics. The « path » of integration in our action principle is analogous to the path of a particle in classical mechanics. In the latter case the path is not endowed with any intrinsic properties. Restriction II makes exactly this same assumption about the « path » of integration which is space-time.

The example of Section 5 is the simplest one that we have been able to construct. There is no theory corresponding to a single scalar field. Boson theory can be constructed, but involves at least 2 scalar and 2 vector mesons. This is mainly due to Restriction II, but also reflects the extra symmetries introduced by the 6-dimensional formalism. Rotations in the normal plane correspond to an intrinsic symmetry similar to isotopic spin.

The picture of space-time as a surface in 6-space is similar to the physics of a soap-film, except that we have ruled out the possibility of a surface tension. If Restriction II can be shown to be too severe, the re-introduction of surface tension is probably the most plausible generalization. This would ensure the flatness of empty space, and would probably facilitate quantization. The theory as it now stands contains no free gravitational field, and a canonical quantization is not at all obvious. If Schwinger's action principle ([15]) is really an action principle for quantized fields, the matter field may be quantized in the flat 6-space, but the quantization of the « fields » $Z^{\lambda}(x)$ remains obscure. A completely different possibility is that the present theory stands in the same relation to a quantized theory as does classical mechanics to quantum theory. The question of quantization is not the most pressing one, however. Of more immediate interest is the comparison with general relativity. This is obstructed by the non-existence of a theory of a single scalar field. The construction of examples must necessarily begin with an $L^{\alpha\beta}$. Hence the simplest way to compare with general relativity is through the example of Section 5. Work on this is in progress. In the meantime it is encouraging that the path equation excludes the formation of wave-packets in flat space, at least in this example.

([15]) J. Schwinger: *Phys. Rev.*, **82**, 914 (1951).

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It is a pleasure to thank Dr. V. GLASER for many helpful and stimulating discussions. Conversation with Professors M. FIERZ, W. THIRRING and J. M. JAUCH are also greatfully acknowledged, as well as the hospitality of CERN.

RIASSUNTO (*)

Nel tentativo di creare un punto di contatto fra la teoria del campo e la relatività generale si propone di studiare gli spazi-tempi contenibili negli spazi piani a maggior numero di dimensioni come passi intermedi fra la relatività speciale e generale. Si trova che la teoria del campo di una singola particella formulata per mezzo del principio dell'azione invariante di Lorentz è suscettibile di una estensione molto naturale e semplice agli spazi-tempo contenibili in uno spazio piano a sei dimensioni. Considerando lo spazio-tempo come una quadrisuperficie nello spazio a sei dimensioni, si determinano le proprietà metriche del primo richiedendo che l'azione sia stazionaria rispetto alle variazioni della quadrisuperficie. Si ottiene una teoria più esatta imponendo alcune restrizioni piuttosto severe alla forma del lagrangiano. Si studia dettagliatamente un esempio semplice e si trova che esso descrive una coppia di campi di Dirac interagenti attraverso il campo gravitazionale. Il principale pregio della teoria è che essa prende in considerazione in maniera coerente le mutue interazioni fra materia e metrica. Appare plausibile che sia contenuto in essa il principio di Mach. Non si esegue la seconda quantizzazione, tuttavia essa può essere eseguita abbastanza semplicemente ricorrendo al metodo di Schwinger.

(*) *Traduzione a cura della Redazione.*

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Ergodic Theorem in Quantum Mechanics.

Evaluation of the probability of an exceptional initial condition.

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Summary. — In this paper an evaluation is made of the probability of finding an exceptional initial condition in the sense of the Bocchieri-Loinger formulation of the quantum ergodic theorem. The result is compared with that obtained by Pauli and Fierz who made a corresponding evaluation of the probability of finding a non-thermodynamical macro-observer in the original von Neumann formulation.

Introduction.

The difficulties and the unsatisfactory aspects of the classic von Neumann approach (1) (see also (3-7)) to the quantum ergodic theorem have recently been overcome in a new formulation given by Bocchieri and Loinger (2) (*).

The fundamental result of the above mentioned authors is expressed by the following relation:

$$\mathscr{A}\, M \sum_{\nu=1}^{N} \frac{S^2}{s_\nu^2}\left(u_\nu(t) - \frac{s_\nu}{S}\right)^2 < \sum_{\nu=1}^{N} \frac{1}{s_\nu} \ll 1\,, \tag{1}$$

(1) J. von Neumann: *Zeits. f. Phys.*, **57**, 30 (1929).
(2) P. Bocchieri and A. Loinger: *Ergodic Foundations of Quantum Statistical Mechanics*, *Phys. Rev.* **114**, no. 4 (1959).
(3) W. Pauli and M. Fierz: *Zeits. f. Phys.*, **106**, 572 (1937).
(4) M. Fierz: *Helv. Phys. Acta*, **28**, 705 (1955).
(5) I. E. Farquhar and P. T. Landsberg: *Proc. Roy. Soc.*, A **239**, 134 (1957).
(6) P. Bocchieri and A. Loinger: *Phys. Rev.*, **111**, 668 (1958).
(7) G. Ludwig: *Zeits. f. Phys.*, **150**, 346 (1958); **152**, 98 (1958).
(*) This paper will be referred to in the following as BL.

where a certain subdivision of the energy shell into cells is assumed to be given (S and s_ν are the numbers of eigenstates belonging to the energy shell and to the ν-th cell respectively), $u_\nu(t)$ is the probability of the state vector of the system lying in the ν-th cell; M denotes the time average and $\mathscr{A}$ the average over the initial conditions.

The average $\mathscr{A}$ is to be performed on the basis of a postulate of equiprobability for all the initial conditions: this averaging procedure substitutes the von Neumann average over the macro observers which is physically unfounded.

The quantities

$$\Delta_\nu(t) = \frac{(u_\nu(t) - s_\nu/S)^2}{s_\nu^2/S^2}$$

being all positive, relation (1) allows one to conclude that they remain for the greater part of the time $\ll 1$ for the overwhelming majority of the initial conditions, that is to say that the relative square deviation of the $u_\nu(t)$'s from their respective microcanonical s_ν/S is very small.

In this paper we intend to deduce a stronger relation than (1), more precisely we intend to give an explicit evaluation (majoration) of the probability of finding an initial condition in correspondence with which even one of the quantities $\Delta_\nu(t)$ alone remain greater than a fixed value a for a time fraction greater than a given $\bar{\vartheta}$.

1. – Evaluation of the probability $P\{\Delta_\nu(t) > a\}$ at a given time.

We want to evaluate in this section the probability of the quantity $\Delta_\nu(t)$, at a given time t, being greater than a.

To this end first of all we recall here the precise mathematical formulation of the postulate of equiprobability of the initial conditions as implied in BL.

Let $\varphi_1, \varphi_2, \ldots, \varphi_S$ be a system of basic vectors of the energy shell; the initial state may be written in the following way:

$$\psi_0 = \sum_{\varrho=1}^{S} c_\varrho \varphi_\varrho;$$

it is thus characterized by a vector $(c_1, c_2, \ldots, c_S)$ of a unitary S dimensional space (or equivalently, if we put $c_\varrho = x_\varrho + iy_\varrho$, by a vector $(x_1 y_1, \ldots, x_S y_S)$ of an euclidean $2S$ dimensional space), whose extremity lies on an hypersphere of equation $\sum_{\varrho=1}^{S} |c_\varrho|^2 = 1$: the postulate in question consists in attributing to a

set of initial states which corresponds to a certain region of the hypersphere a statistical weight proportional to the area of that region. The evaluation of the probability of a quantity—which depends on the initial conditions—having a certain value is thus reduced to a purely geometrical procedure analogous to that employed by von Neumann.

In particular the probability that the quantity

$$u = \sum_{\varrho=1}^{s} |c_\varrho|^2 \tag{2}$$

has a value between u and $u+\mathrm{d}u$ is given by [1]:

$$W_s(u)\,\mathrm{d}u = \frac{(S-1)!}{(s-1)(S-s-1)!}\, u^{s-1}(1-u)^{S-s-1}\,\mathrm{d}u\,.$$

This expression is valid independently of the choice of basic vectors, and in particular of the order in which they are taken. If we now denote by $\omega_{\nu i}^{(0)}$ ($\nu = 1, 2, \ldots, N$; $i = 1, 2, \ldots, s_\nu$; $\sum_{\nu=1}^{N} s_\nu = S$) the basis of the energy shell relative to a subdivision of it in cells, and by $\psi_t = \exp[-(i/\hbar)Ht]\psi_0$ the state vector of the system at time t, we may write (cfr. BL):

$$u_\nu(t) = \sum_{i=1}^{s_\nu} |(\omega_{\nu i}^{(0)}, \psi_t)|^2 = \sum_{i=1}^{s_\nu} |(\omega_{\nu i}^{(t)}, \psi_0)|^2\,, \tag{3}$$

where

$$\omega_{\nu i}^{(t)} = \exp\left[\frac{i}{\hbar} Ht\right] \omega_{\nu i}^{(0)}\,; \tag{4}$$

then (as the transformation (4) is unitary) the last term of (3) is of form (2), so we have:

$$P\{\Delta_\nu(t) > a\} = \left(\int_0^{s_\nu(1-\sqrt{a})} + \int_{s_\nu(1+\sqrt{a})}^{1} \right) W_{s_\nu}(u)\,\mathrm{d}u\,.$$

The quantity appearing in the right hand side of this equation, apart from the substitution $a \to (s_\nu/S)a$, is studied in a different context by Pauli and Fierz [3]; making use of their result we have:

$$P\{\Delta_\nu(t) > a\} < \exp[-\varkappa\sqrt{s_\nu a} + \varkappa + \log S]\,, \quad \varkappa = 1 - \log 2 \cong 0.3\,. \tag{5}$$

This is the evaluation which we wished to obtain for $P\{\Delta_\nu(t) > a\}$. A first interesting relation can be deduced from this and i is as follows:

$$(6) \qquad P\{\bigcup_{\nu=1}^{N}(\Delta_\nu(t) > a)\} \leqslant \sum_{\nu=1}^{N} P\{\Delta_\nu(t) > a\} < \exp[-\varkappa\sqrt{s_{\min} a} + \varkappa + \log S + \log N]$$

it gives the probability that ata fixed time t, even one of the $\Delta_\nu(t)$ aloneis greater than a.

2. – An elementary property of the functions of two stochastic variables.

Let τ and λ be two stochastic variables uniformly distributed and normalized: $\int d\tau = \int d\lambda = 1$. If $x(\tau, \lambda)$ is a continuous function of these variables, the probabilities that the inequality

$$x(\tau, \lambda) > \xi$$

be satisfied, for a given value $\bar{\tau}$ of τ or $\bar{\lambda}$ of λ or with no restriction on the possible values of τ and λ, are given by:

$$(7) \qquad \begin{cases} P\{x(\bar{\tau}, \lambda) > \xi\} = \int\limits_{I(\bar{\tau}, \xi)} d\lambda = U(\bar{\tau}, \xi)\,, \\ P\{x(\tau, \bar{\lambda}) > \xi\} = \int\limits_{J(\bar{\lambda}, \xi)} d\tau = V(\bar{\lambda}, \xi)\,, \\ P\{x(\tau, \lambda) > \xi\} = \int d\tau \int\limits_{I(\tau, \xi)} d\lambda = \int d\lambda \int\limits_{J(\lambda, \xi)} d\tau\,, \end{cases}$$

where $I(\bar{\tau}, \xi)$ and $J(\bar{\lambda}, \xi)$ are those sets of values of λ and τ for which one has $x(\bar{\tau}, \lambda) > \xi$ and $x(\tau, \bar{\lambda}) > \xi$, respectively. From (7) one has immediately

$$(8) \qquad \int d\lambda\, V(\lambda, \xi) = \int d\tau\, U(\tau, \xi)\,,$$

which is the property we wanted to recall.

3. – Evaluation of the probability $P\{\bigcup_{\nu=1}^{N}[\vartheta_\nu(a, \psi_0) > \bar{\vartheta}]\}$.

If in the previous paragraph we identify λ with the set of variables which characterize the state ψ_0 and put $\tau = t/T$, where $(0, T)$ is the time interval of interest (finite and otherwise arbitrary), we may apply the property there recalled to the quantities $\Delta_\nu(t)$.

More precisely $U(\tau, a)$ becomes the probability that for $t=\tau T$, $\Delta_\nu(t)$ is greater than a, $V(\lambda, a)$ (which will be denoted here more explicitly by $\vartheta_\nu(a, \psi_0)$) becomes the fraction of the time interval of interest during which $\Delta_\nu(t)$ is greater than a.

From (8) and the results of Section **1** (notice that the evaluation there given for $U(\tau, a)$ is actually time independent), we obtain:

$$\mathcal{A}\vartheta_\nu(a, \psi_0) < \exp[-\varkappa\sqrt{as_\nu} + \varkappa + \log S],$$

from which:

$$P\{\vartheta_\nu(a, \psi_0) > \bar{\vartheta}\} < \exp[-\varkappa\sqrt{as_\nu} + \varkappa + \log S - \log\bar{\vartheta}]. \tag{9}$$

Our final result can now be easily obtained in a way similar to that used in Section **1**, in connection with eq. (6):

$$P\{\bigcup_{\nu=1}^{N}[\vartheta_\nu(a, \psi_0) > \bar{\vartheta}]\} < \exp[-\varkappa\sqrt{as_{\min}} + \varkappa + \log S + \log N - \log\bar{\vartheta}]. \tag{10}$$

Obviously this relation has a meaning only if $a \gg 1/s_{\min}$.

If the following condition:

$$s_{\min} \gg \log S \tag{11}$$

is satisfied, relation (10) assures that the probability of even one of the $\vartheta_\nu(a, \psi_0)$ alone being greater than $\bar{\vartheta}$ is negligibly small for appreciable values of a, when $\bar{\vartheta} \gg 1/S$.

Analogous considerations hold for relation (6).

4. – Concluding remarks.

To conclude we would like to point out that values physically reasonable for s_ν and N are, for instance (see (4)), $s_\nu = e^{10^{20}}$, $N = e^{10^6}$, condition (11) is then certainly satisfied and the other one $\bar{\vartheta} \gg 1/S$ is perfectly significative.

To facilitate the comparison of our results with those of PAULI and FIERZ we note that, starting from the relation

$$S_\mu - \bar{S}(\psi_t) = -\log\frac{1}{S} + \sum_{\nu=1}^{N} u_\nu(t)\log\frac{u_\nu(t)}{s_\nu} \leqslant \sum_{\nu=1}^{N}{}' \frac{S}{s_\nu}\left(u_\nu(t) - \frac{s_\nu}{S}\right)^2,$$

(S_μ is the microcanonical and $\bar{S}(\psi_t)$ is the von Neumann entropy of the system)

and taking the others:

$$P\left\{\frac{S}{s_\nu}\left(u_\nu(t)-\frac{s_\nu}{S}\right)^2\right\}>a<\exp\left[-\varkappa\sqrt{Sa}+\varkappa+\log S\right],$$

$$P\left\{\sum_{\nu=1}^{N}\frac{S}{s_\nu}\left(u_\nu(t)-\frac{s_\nu}{S}\right)^2>a\right\}<P\left\{\bigcup_{\nu=1}^{N}\left[\frac{S}{s_\nu}\left(u_\nu(t)-\frac{s_\nu}{S}\right)^2>\frac{a}{N}\right]\right\}<$$

$$<\sum_{\nu=1}^{N}P\left\{\frac{S}{s_\nu}\left(u_\nu(t)-\frac{s_\nu}{S}\right)^2>\frac{a}{N}\right\},$$

into account, with calculations similar to those previously made, we can deduce the following relation directly comparable with (13) of paper [3]:

$$P\{\vartheta(a,\psi_0)>\bar{\vartheta}\}<\exp\left[-\varkappa\sqrt{\frac{S}{N}}\,a+\varkappa+\log S+\log N-\log\bar{\vartheta}\right].$$

The probability that the difference between actual and microcanonical entropy of the system remain greater than a ($a>0$) for a time fraction $\vartheta>\bar{\vartheta}$. is negligibly small for appreciable values of a under the condition:

$$\sqrt{\frac{S}{N}}\gg\log S\,,$$

which is slightly less restrictive than (3).

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We extend our thanks to Profs. CALDIROLA, LOINGER and BOCCHIERI for their kind interest and for stimulating discussions on the subject.

RIASSUNTO

In questo lavoro si dà una valutazione della probabilità di trovare una condizione iniziale eccezionale, nel senso della formulazione di Bocchieri e Loinger del teorema ergodico quantistico. Si confrontano i risultati con quelli ottenuti da PAULI e FIERZ che hanno dato una valutazione della probabilità di trovare un macro-osservatore non termodinamico nella formulazione originale di VON NEUMANN.

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Electromagnetic Processes at High Energies.

Part I. – Anomalous Showers.

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(ricevuto il 15 Giugno 1959)

Summary. — The problem of anomalous showers has been discussed with a view to compare their fluctuations with those for two other samples of electromagnetic cascades. It is concluded that the observed discrepancies result on account of the individuality of the events and that the fluctuations are similar to those for normal cascades. In general the fluctuations are found to be not very much larger than those for a Poisson distribution.

1. – Introduction.

In recent years, the study of electromagnetic processes has assumed a new importance, since some experimental findings indicated marked departures from theory. From observations made in nuclear emulsions, on the development of electron-photon showers and the presence in them of an abnormally large number of tridents, it appeared that the existing theories on cascade development and trident production were inadequate to explain the phenomena at high energies. On account of the many advantages of stripped emulsions over other type of detectors most of the cascade theory problems have been in later years tackled with renewed approach. Investigations at high energies are limited on account of the rarity of such events and so attempts are sometimes made to derive as much information as possible from events found under different experimental conditions. As a result a wide variety of data has accumulated, some indicating accordance with theoretical predictions and other pointing out large discrepancies. In view of the limits of error in-

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volved in the experimental procedure and wide fluctuations allowed by the theory, a straightforward interpretation of the experimental data has not always been possible. In a recent communication ([1]), it was mentioned that during the course of that investigation no shower depicting abnormal behaviour was observed. This conclusion was in fact based upon our having recognized the large fluctuations intrinsic in the nature of the process. However, if we had selected out of our material an individual event and neglected all the others (the situation is somewhat similar to our having accidentally observed only one particular event) it could possibly be classified as an anomalous shower. In order to clarify this point, we propose to discuss in this article the problem of the so-called « Anomalous showers » and consider in that context the various events known to us and try to find out whether the observed anomalies have been apparent or genuine.

2. – Statement of the problem.

In a systematic investigation, the electron-photon showers are normally picked up by scanning for bunches of minimum ionization tracks. Each bunch is then followed through the stack either to the point of materialization of the primary photon or to the point of entry of the electron or electrons into the stack. Such events which may be called isolated showers are the subject of the present discussion. (In some cases, the following back might lead to a high energy disintegration from which the soft component emerges as a result of the decaying high energy π^0-mesons. These showers called related cascades shall not be considered at present.) In most of the cases however a systematic investigation is not carried out with a view to pick up the cascades but during a scanning for some other experiment, a soft cascade is picked up at random. It is evident that in such a case there is a very strong bias in favor of those events being picked up more frequently which have high multiplication either due to very high energy of the primary or to some anomalous processes hitherto unknown. Unless a significant number of these so-called anomalous events are obtained in a systematic investigation, any conclusions on the nature and frequency of these random events might be misleading. The situation is much worse when only a single event is observed and compared with the theoretically predicted behaviour.

Theoretically it is possible to estimate the average number of electrons expected at a certain depth measured with respect to the origin of the shower. A soft shower may be called anomalous if the number function *i.e.*, the distribution of the various electron tracks in regard to the longitudinal development

([1]) P. K. ADITYA: *Nuovo Cimento*, **11**, 546 (1959).

and/or lateral spread are beyond the wide fluctuations allowed by the nature of the process. The observed anomalies can arise possibly from the following sources:

1) indefiniteness of the nature and number of primary particles;

2) uncertainty in energy estimation of the various components of the shower;

3) contribution to the shower development by processes other than the conventional;

4) fluctuations.

2·1. – *a*) *Photon initiated showers*: If the origin of a cascade is a pair of two tracks originating in the emulsion it is natural to assume that the primary has been a photon and the point of materialization is considered as the origin of the cascade. It is however not possible to say without doubt whether the photon has been single or is accompanied by one or more other photons. Because of the conditions of the experiment there exists also a finite probability of a particular shower's existing in the stack in such a way that the track of the parent electron which emitted the photon or photons by the conventional bremsstrahlung process cannot be observed. This probability depends strongly upon the experimental factors, such as the steepness of the event, the clarity of the emulsions and the level of minimum ionization. The single electron might have got highly scattered after a sudden radiation loss of the major part of its energy and appear at such a steep angle as to be easily missed. Failure to detect the single electron in the vicinity of one or more high energy pairs can lead to an event anomalous in regard to either or both of cascade development and lateral spread. Assuming that the experimental conditions are ideal and that the cascade development is found to be much more rapid than that expected for a single photon, or if the lateral spread of the shower is impossible to be expected from a single photon, one is left to assume a larger number of photons being present. These photons could result from the following processes:

i) Up to two photons can be associated with the decay of a π^0-meson, but one meets with the difficulty of finding a nearby source of π^0-mesons. In spite of the time dilatation at very high energies, the π^0-meson because of its very short life time ($\sim 10^{-15}$ s) is not expected to travel large distances. A local source such as a high energy disintegration is not difficult to detect, unless it is a rare collision between a high energy proton (or neutron) and a nucleon of the emulsion, involving production of only neutral particles.

ii) There could be some other unstable particles of long life time decaying into γ-rays. At the present state of knowledge the existence of such a particle is not known.

iii) Two or more photons can be produced in an annihilation process between a proton-antiproton or an electron-positron. CORINALDESI [2] has estimated the probability for p^+-p^- annihilation and in order to explain as many as 20 photons, suggestive of the anomalous event observed by SCHEIN *et al.* [3] needed an energy of $\sim 10^{52}$ eV. In view of the antiprotons being rare, and the impossibility of having energies of that order, the probability of such a process being responsible for the large number of observed anomalous events is negligible. GUPTA [4] and JOSEPH [5] have estimated that in e^+-e^- annihilation at very high energies the probability of multiple photon production of up to 4 or 5 photons is comparable to that for the normal two photon production. Since in cascades, high energy positrons are in equal number as electrons, the remote possibility of a very high energy positron getting annihilated may not be neglected. However, according to HEITLER [6], even at very high energies the positrons are more likely to loose their energy before getting annihilated, so that the contribution to cascade development by this process can be only negligibly small. It may be mentioned that in emulsions, such a process cannot be directly identified, since it is not possible to detect a stopping minimum ionization track and associate it with a cascade that starts developing far ahead. However, if it occurs in an already developed shower it is possible to recognize a track stopping in flight, but the annihilation photons when materializing cannot be distinguished from the normal bremsstrahlung photons. For some events of this kind which show much too large lateral spread of the pairs and for which the probability of having missed the associated electron track or tracks is small, e^+-e^- annihilation seems to be a feasible explanation.

2·1. – *b*) *Electron initiated showers*: If a cascade when followed back leads to a single track at minimum ionization it is most likely to be due to an electron because the cross-section for bremsstrahlung by heavier particles is known to decrease very rapidly with increasing mass. It is at first absolutely essential to follows such a track to its point of entry into the stack, because unless that is done the origin of the cascade cannot be precisely defined. It may be possible to judge the potential range of such a track, if it is not con-

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(2) E. CORINALDESI: *Nuovo Cimento*, **12**, 571 (1954).
(3) M. SCHEIN, D. M. HASKIN and R. G. GLASSER: *Phys. Rev.*, **95**, 855 (1954).
(4) S. N. GUPTA: *Phys. Rev.*, **98**, 1502 (1955).
(5) J. JOSEPH: *Phys. Rev.*, **103**, 481 (1956).
(6) W. HEITLER: *Quantum Theory of Radiation* (Oxford, 1954), p. 271.

venient to follow it back but such a procedure is likely to introduce an unknown bias against some low energy pairs that might have materialized in the unfollowed length and do not show-up now at the point one finds apparently only one track. It is known to every experimentalist that the following back of a single track at minimum ionization involves apart from great care also some uncertainty. Assuming that the single track has been followed to the point of its entry into the stack, it is conventional to assume that point as the origin of the cascade and consider the electron to be alone. This is a reasonable assumption for most of the cases, but might not be so in that rare case when some « bremsstrahlung » has been emitted in the packing material beyond the sensitive volume of the stack or even some created a little outside the stack. Because such photons are likely to materialize at any stage, on or near the primary track these can give rise to an anomalous event with an apparently fast growth and wide aperture. The situation is particularly bad in the case of emulsions, because only a limited portion of the cascade near its origin can be observed. It appears therefore unsafe to derive conclusions on the anomalousness of an event that is initiated by one or more electrons entering the stack from the outside. Considerations as above will apply to a pair of tracks entering the stack (see for example the event due to SCHEIN *et al.* [3]) or even to those events in which the primary photon materializes very soon after entry into the stack.

2·2. – *Uncertainty in energy estimation.* Apart from predicting the average number of electrons expected at a certain depth in the absorber, the cascade theory foresees a certain energy spectrum of the secondary electrons. Both the cascade development and energy spectrum can be specified provided the energy of the primary can be estimated. For most of the anomalous events, the anomalies disappear if the estimated energies are allowed uncertainties in some cases of very large order. It is felt that energies are almost always underestimated. The underestimation of primary energy leads one to classify a cascade as depicting an anomalously rapid growth, while an underestimation of secondary energies gives rise to a discrepancy between the experimentally observed and theoretically predicted energy spectra. The situation is equally bad if different techniques are used for estimating energies in the various energy ranges, so that discrepancies arise in the development, or in the energy spectra or partly in both. At present a number of methods of varying merit are available for energy estimation, with a drawback that not one of them can be employed for the entire region one needs to explore. These are:

i) multiple Coulomb scattering;

ii) opening angle of the pair;

iii) suppression of ionization near pair origin ([7]);

iv) lateral spread of the shower ([8]);

v) longitudinal cascade development.

A detailed consideration of all these methods may preferably be postponed for a later discussion, but a few useful remarks may be made in the context. Primary energies $\geqslant 100$ GeV can be determined equally well by using any of the methods iii), iv) and v) wherever applicable. The first method *i.e.*, suppression of ionization is particularly useful for those events which cannot be followed for sufficient length in the stack. Neither of these methods can be used to estimate energies of the secondary order. The first two methods *i.e.*, multiple scattering and opening angle may in some cases be used for primary pairs of small energy but are suited more for the low energy secondary pairs.

On account of emulsion distortion, microscope stage-noise, personal reading error and above all spurious scattering, energy measurements by multiple scattering at a few GeV or more cannot always be relied upon, whereas a neglect of the multiple scattering while estimating energies from the opening angle of the pair would almost always lead to energy underestimation. From our experience with this work it appears that pair energies can be estimated to a fairly good approximation, if the true opening angle between the two partners is extracted from the observed opening after taking into account the influence of the relative scattering of the tracks ([1,*]).

2·3. – *Processes other than conventional*: In the development of showers, the theory takes into account only the usual emissions by the electron of bremsstrahlung photons and their subsequent materialization. If the growth of the shower is too rapid with respect to that predicted, it is natural to think that apart from conventional processes there might be some contribution due to other phenomena.

It has been known ([9]) that electrons can directly produce a negatron-positron pair in a one step process (in emulsions known as the trident process), but since the cross-section of this process has been theoretically predicted to be negligibly small at the energies commonly dealt with, no account of the trident process has been taken in cascade theory calculations. The experimental observation of a large number of events looking like tridents, initiated a controversy, «whether the trident cross-section is in fact much higher than that

([*]) P. K. ADITYA: *Ind. J. Phys.*, (under publication).
([7]) J. IWADARE: *Phil. Mag.*, **3**, 680 (1958).
([8]) K. PINKAU: *Phil. Mag.*, **2**, 1389 (1957).
([9]) H. J. BHABHA: *Proc. Roy. Soc.*, A **152**, 559 (1935).

predicted ». If so it is possible that such a phenomenon should materially change the shape of the shower at least in the initial stages, the magnitude of the departure increasing with increasing primary energy. Without a critical consideration of the present situation (which it is not our aim to discuss here) it might appear unsafe to make a precise statement. However in view of the probability of bremsstrahlung photons materializing accidentally within the least resolvable distance from electron tracks, there has been growing a general belief that most of the observed tridents are « spurious » and that the number of true tridents is not far from that predicted. Though numerical results of some theoretical attempts ([10,11]) on the inclusion of the trident process into cascade theory are not readily available, it might not be very much wrong to say that the contribution of trident process in the general development of showers (though it may not be completely insignificant) is not appreciable, especially because of the very strong energy dependence of the trident cross-section.

Apart from trident production, which is the strongest rival to the conventional processes, it might appear useful to think of multiple processes such as *a*) production of more than two photons at e^+-e^- annihilation at very high energies; *b*) multiple bremsstrahlung by an electron; *c*) multiple pair production at the materialization of a single photon and *d*) direct production of more than one pair by an electron. Some aspects of process *a*) have been considered before (Section **2**·1, *a*), iii)). For multiple processes one might expect according to HEITLER ([12]) that the cross-section of these processes falls rapidly with n, being proportional to $(\alpha/\pi)^n$, where α is the fine structure constant, $= 1/137$. The multiple processes are therefore not likely to make significant contribution unless such a thing occurs accidentally in the most initial stage of development. For process *b*), GUPTA ([13]) has estimated the cross-section to be negligibly small even at extremely high energies. It may be mentioned that such a process cannot be directly observed under the conditions of our experiments. Isolated examples of multiple pair production by a single photon ([14]) and of the direct production of two pairs by an electron ([15]) have been experimentally observed. It is however, not safe to draw conclusions about the frequency of such events, since only these few events ([14,15]) should not exhaust the whole material ([16]). Such processes though directly observable

([10]) J. W. GARDNER: *Nuovo Cimento*, **7**, 10 (1958).

([11]) S. K. CHAKRABARTY: *Proceedings of the Cosmic Ray Symposium* (Bombay, February 1958), unpublished.

([12]) W. HEITLER: loc. cit., p. 228.

([13]) S. N. GUPTA: *Phys. Rev.*, **99**, 1015 (1955).

([14]) J. E. HOOPER and D. T. KING: *Phil. Mag.*, **41**, 1194 (1950).

([15]) C. CASTAGNOLI and A. MANFREDINI: *Nuovo Cimento*, **8**, 778 (1958).

([16]) A. A. VARFOLOMEEV, R. I. GERASIMOVA and V. A. TUMANYUN: *Žu. Èksper. Teor. Fiz.*, **32**, 969 (1957).

are difficult to be identified without doubt if occurring in already developed cascades, because alternative assumptions such as accidental coincidence are more probable. In view of the large fluctuations permitted in the cascade theory, a separation of the small contribution due to multiple processes is not possible in normal cases.

2·4. – *Fluctuations*: The theoretical formulation of the fluctuation problem in the case of cascades has been known to be hopelessly complex. Since the production of successive electrons is not a typically random process *i.e.*, in which subsequent events are independent of those occurring prior to any one of them, the fluctuations from the average have been expected to be much more than those for a Poisson distribution. Among many theoretical attempts, good qualitative and quantitative arguments are found in the books by HEITLER ([17]) and ROSSI ([18]). ARLEY ([19]) has worked out in detail the nature of the stochastic processes, and in the model proposed by him, under certain circumstances the fluctuations may be as high as the average itself. For our purpose, we have preferred to derive the fluctuations from the little experimental data available to us (for details, see the following section).

3. – Observed anomalous showers.

Experimental data on some of the anomalous events known to us ([3,20-24]) have been presently used to derive the fluctuations. Irrespective of all other considerations (see Appendix), the only criterion adhered to while collecting the events has been to include those showers which originate with a pair materialized within the emulsion or with a closely collimated pair of tracks entering the stack. The development of the « mean cascade » derived from these showers has been plotted in Fig. 1, curve (*a*). The horizontal lines have been drawn to indicate the root mean square deviation of the « distance fluctuation » for the respective pairs. For comparison, plotted in the same figure are the cor-

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([17]) W. HEITLER: loc. cit., p. 394.

([18]) B. ROSSI: *High Energy Particles* (New York, 1952), p. 288.

([19]) N. ARLEY: *Stochastic Processes* (Copenhagen, 1943).

([20]) A. DEBENEDETTI, C. M. GARELLI, L. TALLONE and M. VIGONE: *Nuovo Cimento*, (*a*) **2**, 220 (1955); (*b*) **4**, 1151 (1956).

([21]) L. BARBANTI-SILVA, C. BONACINI, C. DE PIETRI, I. IORI, G. LOVERA, R. PERILLI-FEDELI and A. ROVERI: *Nuovo Cimento*, **3**, 1465 (1956).

([22]) M. MIĘSOWICZ, O. STANICZ and W. WOLTER: *Nuovo Cimento*, **5**, 513 (1957).

([23]) M. KOSHIBA and M. F. KAPLON: *Phys. Rev.*, **100**, 327 (1956), (Shower. (P.-I.).)

([24]) O. B. YOUNG and T. S. YOON: *Phys. Rev.*, **108**, 908 (1957).

responding data from FAY [25] and ADITYA [1] as curves (*b*) and (*c*) respectively. Apart from the probability of there being in some cases (see Appendix) more than one photon at the stage « considered as the origin » of the cascade, the individual fluctuations in curve (*a*) do not seem to be appreciably different from those in curves (*b*) and (*c*). While judging this, it has to be remembered that

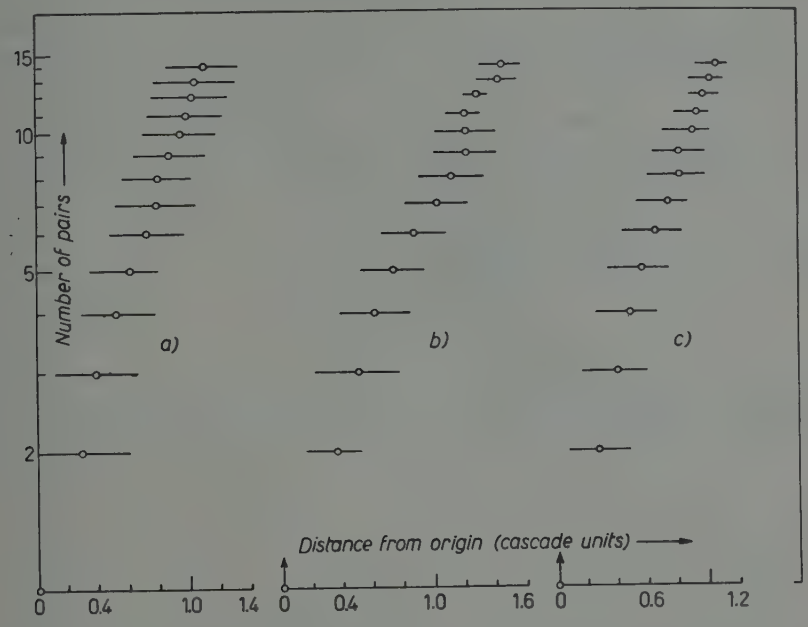

Fig. 1. – Distribution of the distances at which each of the first fourteen pairs materializes, plotted against the number of the pairs. Distances are measured from the origin of the primary pair or from the point of entry into the stack [3]. Curves (*a*), (*b*) and (*c*) refer respectively to the anomalous showers [3,20-24] and to the cascades due to Fay [25] and Aditya [1]. The horizontal lines denote the root mean square deviation of the distances.

for curve (*a*), since the energy of the secondary pairs has not been precisely defined in relation to the energy of the primary, a particular value of the parameter $y = \ln (E_0/E_m)$ cannot be prescribed, whereas in the case of curve (*b*) and (*c*) it is known. This uncertainty is likely to change the slope of the « mean cascade » as well as to allow for larger deviations in the case of anomalous events. Lack of reliable knowledge of the pair energies, does not permit at this stage a consideration of the energy spectra.

In order to be able to find the fluctuations in the number of electrons observed at a certain distance from the origin, data are available from FAY [25]

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[25] H. FAY: *Nuovo Cimento*, **5**, 293 (1957).

and ADITYA [1], In order to find the number of electrons present at a certain depth for the anomalous events, it is necessary to take into account the difference in the number of electrons from twice the number of pairs materializing in that distance. This is due to scattering away of the low energy electrons, and we have used our data to obtain the corresponding approximate values for the anomalous events. The root mean square of the « number fluctuation » has been obtained from the average number observed at various depths and « percentage fluctuation » computed. We have taken in the case of the observed data and for Monte Carlo calculations ([26]):

$$\text{Percentage Fluctuation, (P.F.)} = \frac{100}{N_{\text{mean}}}\left[\frac{\sum_n (N_{\text{observed}} - N_{\text{mean}})^2}{n}\right]^{\frac{1}{2}},$$

and in the case of the Poisson distribution,

$$\text{(P.F.)} = \frac{100}{N}(N-1)^{\frac{1}{2}},$$

where N is the total number of electrons at a certain depth, (or $N-1$ is $\sim$ the number of secondary electrons) and n is the number of events used in

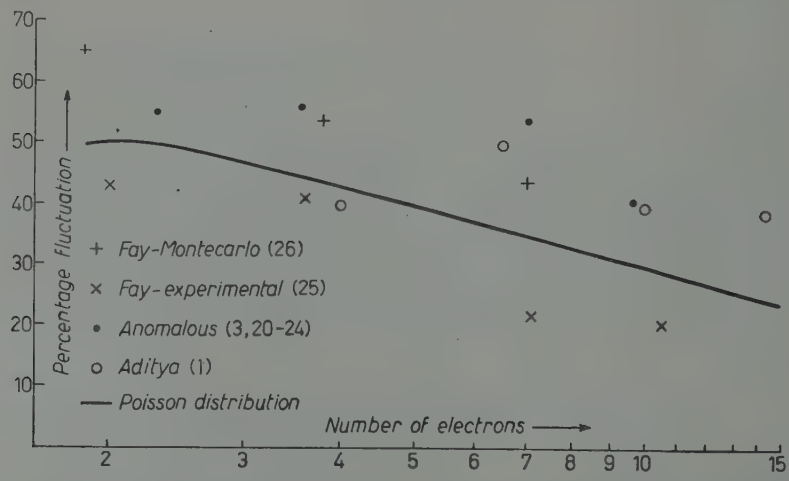

Fig. 2. – Percentage fluctuation (for definition see text), plotted against the total number of electrons observed at a certain depth. The Poisson fluctuation based upon the number of secondary electrons is included for comparison.

a particular sample of cascades. Since P.F., is based upon the standard deviation, one might expect to find one third the number of cascades of a particular sample having fluctuations greater than those given hereafter. We

([26]) H. FAY: *Nuovo Cimento*, **6**, 1516 (1957).

have included in this compilation the results of the Monte Carlo calculations done by FAY [26] and the results are presented in the form of Fig. 2. In this form these may be compared with some of the theoretical computations. It might appear convenient to plot P.F., against the distance from the origin rather than the number of electrons, but since at a certain depth the average number of electrons is a function of the « minimum acceptable energy » and of the theoretical results employed for comparison (see below), we have preferred to plot P.F., against the number of electrons. The « number fluctuations » for the anomalous events do not seem to be significantly different from those for two other samples of cascades [25,1], nor are these very much larger than those for a Poisson distribution, as appears to be the general belief. For example, according to ARLEY [19], P.F., could be $\sim 80\%$ for the region of Fig. 2.

From these considerations we strenghten our pointing out in the begining, that the decision on an individual event being anomalous is hard to make because the fluctuations intrinsic in the process may be large indeed for an individual event.

Before we end this discussion it appears useful to say a word about the « average number of particles » expected at a certain depth, as predicted theoretically. Various authors have compared their experimental findings with some of the many cascade theory calculations, such as of ARLEY [27], BHABHA and CHAKRABARTY [28], BHABHA and HEITLER [29], JÁNOSSY [30], ROSSI and GRIESEN [31] and with the Monte Carlo calculations of FAY [26] and of KAPLON and WILSON [23]. Though the general feature of all these results are similar, in the region of our interest, *i.e.*, initial stages of a cascade, the average number of electrons is not the same, the difference going up to a factor of ~ 1.5. This factor introduces a certain amount of uncertainty in the theoretical interpretation of the experimental data. At this place, it is neither possible nor convenient for us to discuss the various attempts in detail.

4. – Conclusions.

From the foregoing considerations the following conclusions may be summarized:

1) in order to classify an event as anomalous it is essential to make sure of the nature and number of the primary particles. Events originating

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[27] N. ARLEY: *Proc. Roy. Soc.*, A **168**, 519 (1938).
[28] H. J. BHABHA and S. K. CHAKRABARTY: *Phys. Rev.*, **74**, 1352 (1948).
[29] H. J. BHABHA and W. HEITLER: *Proc. Roy. Soc.*, A **159**, 1432 (1937).
[30] L. JÁNOSSY: *Cosmic Rays* (Oxford, 1950).
[31] B. ROSSI and K. GRIESEN: *Rev. Mod. Phys.*, **13**, 240 (1941).

with a single pair are far superior than those initiated by one or more electrons. In order to avoid the observational bias it is useful to select events having smaller angles to the plane of the emulsion;

2) energy estimation should be done by taking into account the multiple Coulomb scattering and the true opening angle derived from the observed aperture;

3) the contribution of direct pair production and of the multiple processes is insignificant in the general development of a shower. In a particular event in which such a process occurs in the very initial stages, it should be possible to identify it;

4) the fluctuations of the so-called anomalous showers are no more than those for the normal cascades. In most of the cases the fluctuations are not very much larger than those for a Poisson distribution. In any particular case, however, the deviations may be as large as the average itself;

5) the average number of electrons expected at a certain depth is not known very precisely, as various theoretical results are not identical. Before the electronic computers can be used for this purpose, it is rather hard to predict accurately both the number distribution and the number fluctuation.

* * *

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APPENDIX

The following comments on some of the anomalous events may be made.

Schein et al. ([3]) *event:* If one of the first two tracks is the parent electron and the other is due to an electron from a high energy high disparity pair

created outside the sensitive volume of the stack, then the origin of the cascade cannot be defined. In that case both the longitudinal development and the lateral spread could be in accordance with a normal cascade. But for the lateral spread, the longitudinal development is not beyond the fluctuations for a conventional cascade.

Shower P-I ([23]): As pointed out by the authors, the anomalies would disappear provided the primary energy has been underestimated. Since energies were obtained from the opening angle without taking into account the contribution due to multiple relative scattering, the fears are likely to be true, especially since Borsellino's relation was used. (The energies in the case of Borsellino's relation are smaller by a factor of ~ 5 as compared to those obtained from the relation due to Stearns ([32])). The possibility of multiphoton origin may not be excluded in view of the presence of another pair similar to the primary pair, at a radial distance of 5.3 mm.

First event ([20a]): In view of the steepness of the event (~ 1 mm per emulsion), it is not improbable that it has been impossible to detect the parent electron, especially if in a radiation process it had been scattered through a large angle. In that case since the origin of the shower would most probably be the point of entrance of the « imaginary electron » into the stack, it might not be difficult to explain both the fast longitudinal growth and the unusual lateral spread.

Second event ([20a]): The presence of the low energy pair prior to the high energy one suggests definitely that the high energy photon was not alone at the point « considered as the origin of the cascade ». As the latter pairs all arise after about one cascade unit (which may be nothing else but fluctuation) from the first two pairs, it is improbable that more than two photons were present. The steepness of the event again allows the possibility of the parent electron having passed undetected. FAY ([26]) has carried out Monte Carlo calculations on fifteen showers of high energy and obtained the resemblance of one of the computed showers with this event under reference.

([32]) M. STEARNS: *Phys. Rev.*, **76**, 836 (1949).

RIASSUNTO (*)

È stato discusso il problema degli sciami anomali allo scopo di mettere a raffronto le fluttuazioni degli sciami stessi con quelle di due altri tipi di cascate elettromagnetiche. Si conclude che le discrepanze osservate sono dovute alla individualità degli eventi, e che le fluttuazioni sono simili a quelle delle normali cascate. Si trova che, in linea generale, le fluttuazioni sono molto più ampie di quelle dovute a una distribuzione di Poisson.

(*) *Traduzione a cura della Redazione.*

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Lepton Conservation and Time Reversal in β-decay.

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Summary. — It was investigated which conclusions can be inferred from β-decay experiments taking into account experimental errors but without making theoretical assumptions.

Prior to the time of parity violation in β-decay it was thought that the only way to test the conservation of leptons was the double β-decay or the inverse β-decay. However, discussing the theoretical results obtained by Pauli [1], Kahana and Pursey [2] and Lüders [3] in the light of the recent experiments it becomes evident that lepton conservation can be checked only in single β-decay experiments (*). Furthermore it can be shown that the negative result found in ordinary time reversal experiments allows no conclusion about time reversal invariance as long as maximum breakdown of parity or conservation of lepton charge has not been established.

1. – General considerations.

The discussion of β-decay is usually based upon the interaction density (**)

$$(1)\qquad \begin{cases} \mathcal{H} = \sum_{i,\lambda} \bar{\psi}_p O_i \psi_N [\bar{\psi}_e O_i (C_i^\lambda \psi_\alpha + D_i^\lambda \gamma_5 \psi_\alpha^C] + \text{h. c.} \\ i = S, V, T, A, P \qquad \psi_R = (1-\gamma_5)\psi_\nu \qquad \psi_R^C = (1-\gamma_5)\psi_\nu^C , \\ \alpha = R, L \qquad\qquad \psi_L = (1+\gamma_5)\psi_\nu \qquad \psi_L^C = (1+\gamma_5)\psi_\nu^C , \end{cases}$$

(1) W. Pauli: *Nuovo Cimento*, **6**, 204 (1957).

(2) S. Kahana and D. L. Pursey: *Nuovo Cimento*, **6**, 1469 (1957).

(3) G. Lüders: *Nuovo Cimento*, **7**, 171 (1958).

(*) We shall not consider here the decay of mesons which might involve different kinds of interactions.

(**) We use the notation of Kahana and Pursey (2).

where local interaction is assumed but neither conservation of parity nor lepton charge, ψ_R and ψ_L are right-handed and left-handed neutrino states.

Assuming a vanishing mass of the neutrino PURSEY (4) and PAULI (1) showed that a five parameter transformation can be applied to the coupling constants

$$(2)\qquad \begin{cases} \hat{\xi}_i = \xi_i \cdot A \cdot \exp[i(\varphi+\theta)], & A = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}, \\ \hat{\eta}_i = \eta_i \cdot A \cdot \exp[i(\varphi-\theta)], & |A| = 1, \\ \xi_i = (C_i^L, D_i^L), \quad \eta_i = (C_i^R, D_i^R). \end{cases}$$

Physical results are not affected by this transformation and, therefore, experimental results can depend only upon invariant combinations of coupling constants.

A complete set of invariants is given by

$$(3)\qquad \begin{cases} K_{ij} = K_{ji}^* = \frac{1}{2}(C_j^\gamma C_k^{\gamma *} + C_j^L C_k^{L*} + D_j^R D_k^{R*} + D_j^L D_k^{L*}), \\ L_{ij} = L_{ji}^* = \frac{1}{2}(-C_j^\gamma C_k^{\gamma *} + C_j^L C_k^{L*} - D_j^R D_k^{R*} + D_j^L D_k^{L*}), \\ I_{jk}^* = I_{kj}^* = \frac{1}{2}(C_j^L D_k^R + C_k^L D_j^R - C_k^\gamma D_j^L - C_j^R D_k^L), \\ -J_{jk}^* = J_{kj}^* = \frac{1}{2}(-C_j^L D_k^R + C_k^L D_j^\gamma + C_k^R D_j^L - C_j^R D_k^L). \end{cases}$$

All phenomena of single β-decay can be expressed with the help of K_{ij} and L_{ij}, whereas the double processes depend on combinations of I_{ij} and J_{ij}, *e.g.* $I_{ij}\, I_{lm}^*$. If the neutrino mass is non-zero, additional invariants K_{ij}' and L_{ij}' will appear (5). Only 35 of these invariants are independent as there exist many identities among them.

KAHANA and PURSEY (2) and LÜDERS (3) showed that for special cases of interest, general conditions can be formulated with the help of these invariants.

It will be shown here, that these conditions can be interpreted geometrically thus enabling us to find out the interrelationship between the different cases without algebraic operations and to discuss the experimental results in a simpler way.

The conditions for the invariants imply certain restrictions for the coupling constants. With the help of the transformations (2) the general form of these restrictions can be transformed into equivalent « normal forms », which are simple geometrical relations between the vectors ξ_i and η_i defined in a (C, D)-plane. These normal forms are listed in Table I.

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(4) D. L. PURSEY: *Nuovo Cimento*, **6**, 266 (1957).
(5) C. P. ENZ: *Nuovo Cimento*, **6**, 250 (1957).

TABLE I.

	Condition for invariants	Normal forms
A) Complete polarization of neutrino	$K_{ii} = L_{ii}$ or $K_{ii} = -L_{ii}$ (same sign for all i)	$\xi_i = 0$ or $\eta_i = 0$ (for all i)
B) Maximum interference	$(K_{ij} \pm L_{ij})(K_{kl} \pm L_{kl}) = (K_{il} \pm L_{il})(K_{kj} \pm L_{kj})$	$\eta_i \parallel \eta_j$ and $\xi_i \parallel \xi_j$
C) Lepton conservation	$I_{ij} = J_{ij} = 0$ (and B if A holds)	$\eta_i \parallel \xi_j$ (and $\eta_i \parallel \eta_j$, $\xi_i \parallel \xi_j$ if A holds)
D) Two state theory	$(K_{ij}+L_{ij})(K_{kl}-L_{kl}) = (I^*_{ik}+J^*_{ik})(I_{jl}+J_{jl})$ (and B if A holds)	$\eta_i \perp \xi_j$ (and $\eta_i \parallel \eta_j$, $\xi_i \parallel \xi_j$ if A holds)
I) Parity conservation	$L_{ij} = 0$, $I_{ij} J^*_{kl} = 0$	$\eta_i = \pm \xi_i$ or $\eta_i \mid \xi_i$
II) C-invariance	$\mathrm{Im}\, K_{ij} = L_{ij} = \mathrm{Im}\, I_{ij} I^*_{kl} = I_{ij} J_{kl} = \mathrm{Im}\, J_{ij} J^*_{kl} = 0$	$\eta_i = \pm \xi^*_i$ or $\eta_i \mid \xi^*_i$
III) T-invariance	$\mathrm{Im}\, K_{ij} = \mathrm{Im}\, L_{ij} = \mathrm{Im}\, I_{ij} I^*_{kl} = \mathrm{Im}\, I_{ij} J^*_{kl} = \mathrm{Im}\, J_{ij} J^*_{kl} = 0$	no necessary condition. ξ_i, η_i = real is sufficient

The following signs were used: $\parallel$ for parallel or antiparallel, $\perp$ for orthogonal, $\mid$ for mirror image with respect to the C- or D-axis. (*e.g.* $\eta_i \mid \xi_i$ means $C^R_i = C^L_i$; $D^R_i = -D^L_i$ or $C^R_i = -C^L_i$; $D^R_i = D^L_i$.)

We shall add a few remarks to Table I and show that with the help of the normal forms a better understanding of the conditions for the invariants is possible.

Previous authors ([2,3]) defined case A (cfr. Table I) by the relation $K_{ij} = \pm L_{ij}$ (same sign for all i, j). However, from $\eta_i = 0$ or $\xi_i = 0$ it follows immediately that this equation is satisfied automatically for $i \neq j$ if it holds for $i = j$. Physically this means that if the uncharged lepton is completely polarized in a pure decay (involving one interaction only) it will have the same polarization if emitted in a mixed transition with interference of two interactions.

The physical meaning of case B cannot be recognized from the equations involving the K_{ij} and L_{ij}. From the normal form $\eta_i \parallel \eta_j$ and $\xi_i \parallel \xi_j$, however, it is obvious that case B is equivalent to maximum interference between different interactions. The parallelism of the vectors η_i and ξ_j implies that $\xi^*_i \xi_j$ and $\eta^*_i \eta_j$ have the highest possible values.

In addition one notices that case A and B are complementary in so far as A gives a condition for $i = j$ whereas B is non trivial only for $i \neq j$.

The general conditions for case C and D are trivially satisfied if A holds. The physical reason for this relation is easily understood. If $\eta_i = 0$ or $\xi_i = 0$ the neutrinos and antineutrinos in β^--decay are completely polarized and their helicities are the same (*). Thus a double β-decay without emission of neutrinos or antineutrinos is forbidden as the particle emitted in the first transition cannot be absorbed in the second transition on account of its wrong helicity.

As a consequence double β-decay will not occur no matter whether lepton charge is conserved or not. In the case of complete neutrino polarization lepton charge is conserved if $\eta_i \| \eta_j$ and $\xi_i \| \xi_j$. This means that in a mixed transition (induced by two or more interactions) a pure neutrino or antineutrino is emitted and not a mixture of these two particles. This can be checked experimentally only by observing single β-decay processes.

As far as T-invariance is concerned there is no normal form which is sufficient and necessary. Usually it is thought that in case of T-invariance all the coupling constants must have a common phase factor. If this is true they can be transformed by (2) into real numbers. However, this is not a necessary condition. $C_i^L = \pm D_i^L =$ real and $C_j^L = \mp D_j^L =$ imaginary; $C_i^R = D_i^R = C_j^R = D_j^R = 0$ (at least one $i \neq j$) is an example of an interaction which is invariant with respect to time reversal but not all the coupling constants can be made real at the same time.

Furthermore, as can be inferred from condition III (Table I) it would be necessary in general to perform single *and* double β-decay experiments in order to test time reversal, as it might happen that K_{ij} and L_{ij} are real but not all of the products $I_{ij}I_{kl}^*$, $I_{ij}J_{kl}^*$ and $J_{ij}J_{kl}^*$ (**). On the other hand in the case of completely polarized neutrinos only single β-decay experiments are suitable in order to check time reversal.

Relations between the special cases listed in Table I can be obtained without algebraic operations simply by comparing the normal forms:

1) All the single conditions A, B, C, D, I, II and III are independent. *E.g.* a two state theory does not imply the complete neutrino polarization. But there are relations if one considers more than two cases at the same time. A well known example is the PCT-theorem which connects I, II and III.

(*) If in β^--decay neutrinos and antineutrinos with negative helicity are emitted ($C_i^L \nu_L + D_i^L \bar{\nu}_L$) then uncharged leptons with positive helicity ($C_i \nu_R + D_i \bar{\nu}_R$) are emitted in β^+-decay.

(**) As K_{ii}, L_{ii} and $I_{ii}I_{ii}^*$ are real and $J_{ii} = 0$ invariance with respect to time reversal can be checked only in mixed transitions ($i \neq j$).

2) If A is excluded, C and D imply B.

3) If case A holds the general condition for C and D is satisfied trivially.

4) Case A and B taken together imply both cases C and D, and viceversa. In addition it is seen that violation of P and C-invariance is maximum.

5) Further relations can be found. *E.g.* a two state theory can conserve parity only if $\eta_i \parallel \xi_i$ ($\eta_i = \pm \xi_i$ is excluded because $\eta_i \perp \xi_i$). In this case lepton charge cannot be conserved as this would require η_i and ξ_i to be parallel.

2. – Discussion of experiments.

Usually the experimental results are discussed in the literature on the basis of certain assumptions, *e.g.* the two component theory with lepton conservation. We shall summarize here the information that can be gathered from the experiments *without making idealizing assumptions*. Only allowed β-decay experiments will be considered, neglecting Coulomb correction terms which are too small to be detected.

2·1. *ft-values.* – From measurements [6] of the *ft*-values of 0-0-transitions (^{14}O, ^{26}Al, ^{34}Cl) one obtaines

$$\sqrt{K_{VV} + K_{SS}} = (1.410 \pm 0.009) \cdot 10^{-49} \text{ erg cm}^3 .$$

The recently measured *ft*-value of the neutron [6] gives

$$R = \frac{K_{AA} + K_{TT}}{K_{VV} + K_{SS}} = 1.45 \pm 0.08 .$$

2·2. *Recoil experiments.* – From these experiment information on the type of interaction can be obtained. Recent measurements [6,7] show that

$$K_{TT}/K_{AA} < 0.07 \qquad \text{and} \qquad K_{SS}/K_{VV} < 0.07 .$$

This implies $\xi_{T,S} \approx 0$ and $\eta_{T,S} \approx 0$ and therefore also L_{ij}/K_{AA}, I_{ij}/K_{AA} and J_{ij}/K_{AA} ($i, j = S, T$) are close to zero with an error of the same size or even smaller as that of K_{TT}/K_{AA} or K_{SS}/K_{VV}, whereas *e.g.* $K_{TA}/K_{AA} < 0.26$.

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[6] Compare M. GOLDHABER: *Annual Intern. Conf. on High Energy Physics at CERN* (1958).

[7] W. B. HERRMANNSFELDT, R. L. BURMAN, P. STÄHELIN and J. S. ALLEN and T. H. BRAID: *Phys. Rev. Lett.*, **1**, 61 (1958); A. H. SNELL: *Conference on Weak Interactions at Gatlinburg* (1958).

2·3. *Parity experiments with pure* β*-transitions.* – Measuring the electron distribution from oriented nuclei ([8]), the β-γ circular polarization correlation ([9]) and the electron polarization ([10]) for pure Gamow-Teller or Fermi-transitions one can show that

$$\frac{L_{TT}-L_{AA}}{K_{TT}+K_{AA}}=-1\pm 0.04 \qquad \text{and} \qquad \frac{L_{SS}-L_{VV}}{K_{SS}+K_{VV}}=-1\pm 0.04\,.$$

These relations can be satisfied only if

$$L_{TT}=-K_{TT}(1\pm 0{,}04) \qquad \text{and} \qquad L_{AA}=+K_{AA}(1\pm 0.04)\,,$$

$$L_{SS}=-L_{SS}(1\pm 0.04) \qquad \text{and} \qquad L_{VV}=+K_{VV}(1\pm 0.04)\,,$$

which implies maximum violation of parity.

The different signs for T and A (and S and V, respectively) mean that the neutrinos emitted in these two interactions have opposite helicities.

The recoil experiments, however, suggest that $L_{TT}=K_{TT}=L_{SS}=K_{SS}=0$. If this idealization is made, one obtains the case of completely polarized neutrinos (case A). As was shown above this has far reaching consequences for double β-processes.

But even without this assumption it is easy to show that the probability for double β-decay is very small. From the experimental errors quoted in 2) and 3) is follows that

$$I_{ii}I_{ii}^{*}<0.02\,(|K_{ii}|+|L_{ii}|)^2 \qquad \text{and} \qquad I_{TA}<0.26\,K_{AA}\,.$$

Similar relations hold for the other combinations of I_{ij} and J_{kl}.

2·4. *Time reversal experiments.* – The most direct way to test invariance with respect to time reversal is an experiment in which a triple product is observed. Such an experiment has been performed by observing the electron-

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([8]) C. S. Wu, E. Ambler, R. W. Hayward, D. P. Hoppes and R. P. Hudson: *Phys. Rev.*, **105**, 1413 (1957).

([9]) H. Appel, H. Schopper and S. D. Bloom: *Phys. Rev.*, **109**, 2211 (1958); G. Hartwig and W. H. Schopper: *Zeits. f. Phys.*, **152**, 312 (1958).

([10]) H. Bienlein, K. Güthner, H. von Issendorf and H. Wegener: *Conference on Weak Interactions at Gatlinburg* (1958); A. T. Alihanov, G. P. Eliseev and V. A. Ljubimov: *Nucl. Phys.*, **7**, 655 (1958); M. Deutsch, B. Gittelmann, R. W. Bauer, L. Grodzins and A. W. Sunyar: *Phys. Rev.*, **107**, 1733 (1957); F. Boehm, T. B. Novey, C. A. Barnes and B. Stech: *Phys. Rev.*, **108**, 1497 (1957).

neutrino correlation in the decay of polarized neutrons ([11]). The result is

$$D = -\frac{2\,\mathrm{Im}\,(K_{VA} - K_{ST})}{K_{VV} + K_{SS} + 3(K_{AA} + K_{TT})} = -0.04 \pm 0.07\,.$$

From this relation alone, no conclusion about time reversal can be drawn. Combining it with the result of the recoil experiments it follows that $\mathrm{Im}\,K_{VA}/K_{AA} < 0.15$ (*). But even this is not sufficient for time reversal invariance as *all* the conditions III of Table I must be satisfied. A direct test of $\mathrm{Im}\,L_{VA} = 0$ is possible by observing the β-γ correlation from oriented nuclei. However, the experimental difficulties are considerable and no precise answer could be obtained ([12]). A direct check of $\mathrm{Im}\,I_{ij}J^*_{kl} = 0$ is thought to be unfeasible. However, if completely polarized neutrinos are assumed (case A) then $K_{ij} = L_{ij}$ and $I_{ij} = J_{ij} = 0$. In this case the neutron experiment together with the results of recoil experiments proves invariance with respect to time reversal.

2·5. *Test of the interference condition.* Case B can be tested only by combining different experiments. The results of the parity and recoil experiments imply that all the conditions B of Table I are satisfied with an error of only a few percent except the following two

$$|K_{VA}|^2 = K_{VV} \cdot K_{AA} \quad \text{and} \quad |K_{ST}|^2 = K_{SS} \cdot K_{TT}\,.$$

Thus if these two relations can be verified experimentally it is proved that the interference between the interactions is maximum. The necessary information can be inferred from the β-distribution or the β-polarization from oriented nuclei and from the β-γ circular polarization correlation if a mixed β-transition is investigated and if the nuclear matrix elements are known. This is the case for the neutron decay. The asymmetry of electrons emitted from polarized neutrons was measured ([13]) and the result was

$$A = \frac{2R}{1+3R}\left\{\frac{L_{TT} - L_{AA}}{K_{TT} + K_{AA}} + \frac{\mathrm{Re}\,(L_{ST} - L_{VA})}{K_{TT} + K_{AA}}\right\} = -0.09 \pm 0.03\,.$$

([11]) M. T. BURGY, V. E. KROHN, T. B. NOVEY, G. R. RINGO and V. L. TELEGDI: *Phys. Rev. Lett.*, **1**, 324 (1958); M. A. CLARK, J. M. ROBSON and R. NATHANS: *Phys. Rev. Lett.*, **1**, 100 (1958).

([12]) E. AMBLER, R. W. HAYWARD, D. D. HOPPES and R. P. HUDSON: *Phys. Rev.*, **110**, 787 (1958).

(*) This quantity is equal to $R \sin\theta$, where θ is the phase angle between the V and A coupling. It follows $\theta < 10°$.

([13]) M. T. BURGY, V. E. KROHN, T. B. NOVEY, G. R. RINGO and V. L. TELEGDI: **110**, 1214 (1958).

Using the results of the parity experiments and taking into account their experimental errors one obtains the estimate

$$\frac{\mathrm{Re}\,(K_{ST}-K_{VA})}{\sqrt{(K_{TT}+K_{AA})(K_{SS}+K_{VV})}}>0.86\,.$$

This means that this expression is close to its maximum value which is 1. This is possible only if $\eta_S=\lambda\eta_T$ and $\xi_V=-\lambda\xi_A$, where λ is a real number. Deviations from 1 can originate from three independent reasons:

i) K_{ST} and K_{VA} are complex numbers (λ complex). This would be the case if time reversal invariance is violated. Therefore this kind of experiment is usually considered as a test of time reversal. However, a small imaginary part of K_{ST} and K_{VA} would hardly be detectable (*) as $|K_{ij}|^2=(\mathrm{Re}\,K_{ij})^2+(\mathrm{Im}\,K_{ij})^2$. Hence, by assuming λ to be real only a small error will be made in the discussion of this experiment.

ii) $\xi_V/\xi_A=-\eta_S/\eta_T$ does not hold. This condition imposes restrictions on the possible values and phases of the coupling constants. However, violations of this condition are again hardly detectable as contributions from the S and T-interaction are very small.

iii) The most critical condition is the parallelism of ξ_V and ξ_A and of η_S and η_T, respectively. This parallelism is characteristic for a maximum interference and implies $|K_{VA}|^2=K_{VV}\cdot K_{AA}$ and $|K_{ST}|^2=K_{SS}\cdot K_{TT}$. Hence, from the neutron experiment it can be inferred that the first equation is satisfied within an error of about 15%.

As in the case of completely polarized neutrinos lepton conservation (and a two state theory) is equivalent to maximum interference this experiment should be looked upon as a test of lepton conservation.

The experiments discussed in this paper are confirmed very nicely by some other very interesting investigations. But no new or more accurate results are obtained.

We might summarize the results which can be inferred from recent experiments without making any assumptions, in the following way:

1) The β-interaction is mainly A—V. Contributions from S and T are smaller than 10%.

2) Parity violation and polarization of neutrinos is maximum with an error of less than 4%.

(*) Of course, this implies that $\cos\theta$ is observed instead of $\sin\theta$ in an ordinary time reversal experiment. Cfr. (4).

3) The probability for double β-decay without emission of neutrinos is only a few percent of the maximum possible value which is obtained for complete violation of lepton charge conservation.

4) Invariance with respect to time reversal holds with an error of about 15 %.

5) The experiments prove lepton conservation and a two state theory with an uncertainty of about 15 %.

This type of β-interaction is in agreement with several recent treatments of the theory of β-decay ([14]).

([14]) R. P. FEYNMAN and M. GELL-MANN: *Phys. Rev.*, **109**, 193 (1958); E. C. G. SUDARSHAN and R. E. MARSHAK: *Phys. Rev.*, **104**, 1860 (1958); J. J. SAKURAI: *Nuovo Cimento*, **7**, 649 (1958); R. W. THEIS: *Zeits. f. Phys.*, **150**, 590 (1958).

RIASSUNTO (*)

Sono state studiate le conclusioni che si possono dedurre dagli esperimenti sul decadimento β tenendo conto degli errori sperimentali, senza peraltro fare supposizioni teoriche.

(*) *Traduzione a cura della Redazione.*

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Energy Spectra and Angular Distribution of Photoneutrons from Oxygen.

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(ricevuto il 27 Giugno 1959)

Summary. — Energy spectra and angular distribution of photoneutrons from oxygen are investigated by irradiation of a water target with a collimated 31 MeV bremsstrahlung beam. Photoneutrons emitted at θ angles of 30°, 60°, 90°, 120° and 150° with a γ-ray beam are recorded by means of proton recoil tracks in Ilford L-4 400 μm thick photoemulsions. The neutron spectra analysed in 0.5 MeV steps show a fine structure in good agreement with the fine structure found in the proton spectra and in the (γ, n) and (γ, p) cross sections. From these structures many levels of the ^{16}O nucleus are identifyed which are in good agreement with known levels and some levels given with uncertainty are confirmed. For the $^{16}O(\gamma, n)$ reaction the contribution of direct process appears prevalent while for all elements up to this date investigated evaporation accounts for the greater part of emitted neutrons. The ground state transitions amount to at least 60% of the emitted neutrons. The angular distribution of photoneutrons is analysed separately for single neutron peaks and a back and front asymmetry is found in the region of the giant resonance. Results on the $^{16}O(\gamma, n)$ process up to 31 MeV can be interpreted, in good approximation, in terms of *a*) many-level theory for the giant resonance, *b*) single-particle transitions, *c*) charge independence of nuclear forces.

1. – Introduction.

A sistematic study on the photoneutrons from various elements performed in this laboratory shows that although the structure of the photoneutron energy spectra depends on the particular element investigated, the following

general trend may be noted: the energy spectra become harder and the high energy tail is more important as the mass number A decreases [1].

For the heavy elements—Ag [2], Bi [2], Ta [3], Pb [4], Au [5]—the spectra are in general agreement with the statistical evaporation theory with the exception of a high energy tail which accounts for about 10% of the total yield.

For medium elements—Cu [6], Cr [3], Ca [7], Rh [7]—the contribution of the direct process becomes higher.

For Al [1,8] the photoneutron emission occurs mainly by a direct process above 24 MeV photon energy. The lightest element whose neutron spectra are known [1,9] is aluminium.

Although the investigation of photoneutron spectra for light nuclei is somewhat difficult because of the small (γ, n) cross-section, we have decided to study the spectra and the angular distribution of the photoneutrons from oxygen considering the high interest of an investigation of this nucleus.

The level density of ^{16}O is comparatively low since it is double magic and the $^{16}O(\gamma, n)$ process gives the ^{15}O as residual nucleus, which has also a low level density with the first excited state at 5.3 MeV. So after the neutron emission has occurred the residual nucleus may be released either in its ground state or in a state of excitatoin higher than 5.3 MeV.

Another advantage of the investigation of the oxygen nucleus is due to the narrowness of the $^{16}O(\gamma, n)$ cross-section curve, the half width at maximum being only 3.4 MeV according to CARVER and LOKAN [10]. So even if oxygen is irradiated with a bremsstrahlung spectrum, only a relatively narrow part of the γ-spectrum is utilized.

Besides, useful information about the mechanism of photodisintegration can be obtained from the angular distribution of the emitted photoneutrons.

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(1) G. CORTINI, C. MILONE, T. PAPA and R. RINZIVILLO: *Nuovo Cimento*, in press.

(2) G. A. PRICE: *Phys. Rev.*, **93**, 1279 (1954).

(3) G. CORTINI, C. MILONE, A. RUBBINO and F. FERRERO: *Nuovo Cimento*, **9**, 85 (1958).

(4) M. E. TOMS and W. E. STEPHENS: *Phys. Rev.*, **108**, 77 (1957).

(5) S. CAVALLARO, V. EMMA, C. MILONE and A. RUBBINO: *Nuovo Cimento*, **9**, 736 (1958).

(6) P. R. BYERLY and W. E. STEPHENS: *Phys. Rev.*, **81**, 473 (1951).

(7) A. AGODI, S. CAVALLARO, G. CORTINI, V. EMMA, C. MILONE, R. RINZIVILLO, A. RUBBINO and F. FERRERO: *Compt. Rend. Congrès International de Physique Nucleaire*, 7-12 Juillet 1958 (Paris, 1959), p. 625.

(8) F. FERRERO, R. MALVANO, S. MENARDI and O. TERRACINI: *Nucl. Phys.*, **9**, 32 (1958).

(9) M. REVZEN and B. W. SARGENT: *Conference on Photonuclear Reactions* (Washington, 1958).

(10) J. H. CARVER and K. H. LOKAN: *Austr. Journ. Phys.*, **30**, 312 (1957).

2. – Experimental procedure.

2·1. *γ-source.* – The γ-ray beam from the Brown Boveri Betatron of the University of Turin, working at the maximum energy of 31 MeV, was collimated —Fig. 1-*a*—so as to have a diameter of 33 mm at the oxygen target. The water wall screened the recording plates against the spurious neutrons coming from the betatron ([3]).

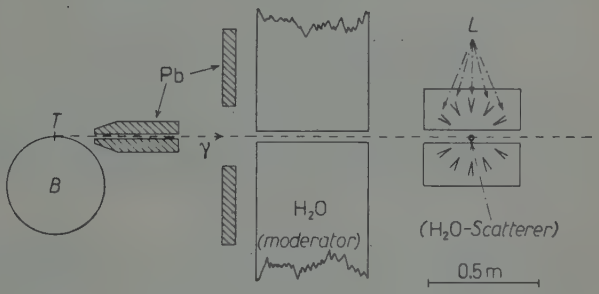
2·2. *Oxygen target,* – The following experimental data for the $^{16}O(\gamma, n)$ process are knwon ([10]):

Threshold energy	15.6	MeV ;
Energy of the cross-section peak	24	MeV ;
Half-width	3.4	MeV ;
Integrated cross-section up to 31 MeV . .	(46 ± 7)	MeV mb .

Taking into account the relatively low integrated cross-section of the $^{16}O(\gamma, n)$ process an oxygen mass of the order of one mole has been irradiated so as to have a reasonable neutron flux on the recording plates.

The irradiation dose was 3 600 röntgen measured by means of an ionization chamber ([11]).

The used target was about one mole of bidi-



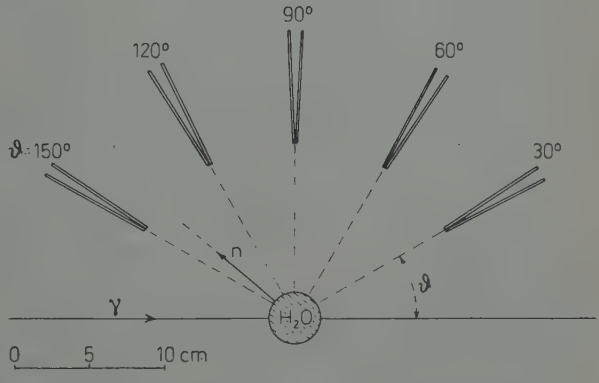


Fig. 1. – *a*) Experimental arrangement: *B* = betatron; *T* = target of the betatron; Pb = lead collimator; H_2O (moderator) = water screen against neutrons from the betatron; H_2O (scatterer) = = target of oxygen; *L* = photoplates. *b*) Experimental arrangement of the photoplates.

([11]) F. Ferrero, R. Malvano and C. Tribuno: *Nuovo Cimento*, **5**, 510 (1957)

stilled water contained in a thin gum vessel whose weight was only 0.15 g.

The presence of hydrogen in the target gives rise to scattering and consequently to the slowing down of the photoneutrons produced in oxgyen. The mean free path of neutrons in H_2O for scattering in H is 5.2 cm for 2 MeV neutrons and 23.2 cm for 15 MeV neutrons, while the mean path of neutrons in the H_2O target was only 1.5 cm. So only a little percentage of photoneutrons produced in H_2O suffer scattering in H.

The little correction which had to be introduced in the observed spectrum to take into account the slowing down of the neutrons in H and in O is discussed in Appendix I.

2·3. *Neutron revelation and scanning of plates.* – The photoneutrons emitted at angles $\theta = 30°$, 60°, 90°, 120° and 150° with the photon beam were detected by means of the proton recoil tracks in Ilford L-4 plates, 400 μm thick (Fig. 1-*b*).

L-4 instead of C-2 plates have been employed because the proton tracks appear more distinctly in the L-4 plates.

Plates of 400 μm instead than 200 μm have been used in order to minimize the probability of escape of the tracks from the emulsion and in order to obtain an advantage on the scanning time, the scanned volume for the scanned surface being double. This gives a gain in the scanning time, the number of proton recoil tracks per unit surface being low.

The method used for scanning the plates with the « slow » and the « fast » method and for analysing the proton recoil tracks are described in a previous work ([3]).

2·4. *Background.* – The water wall was efficient in eliminating spurious neutrons. But as the yield of the $^{16}O(\gamma, n)$ process is low—compared to the yield of heavy elements—the background was not quite negligible specially for 30° and 150° plates whose mean distance from the γ-beam was only 7.5 cm.

Besides, the 30° plates may have accepted, in the region near the γ-beam, some neutrons coming from the Pb collimator.

Therefore, an irradiation was made exposing the plates without target and the background was deduced by the scanning of these plates. Results are given in the following paragraph.

3. – Results.

3·1. *Background.* – For the background determination we have scanned the plates exposed without target.

Results obtained are summarized in Table I.

The background is negligible at the energy $E_n > 5$ MeV, while in the energy region below 5 MeV it is respectively 70, % 15 %, 0, 15 %, 40 % of the acceptable tracks at angles $\theta = 30°$, 60°, 90°, 120°, 150°. Nearly all the background tracks have energy $E_n < 3$ Mev, so that the energy spectra may contain some little error due to the background subtraction in the interval $(3 \div 5)$ MeV while they are quite unaltered for $E_n > 5$ MeV.

An analysis of the tracks with origin in the emulsion but having wrong directions, done in the plates exposed with oxygen target, shows that the background comes preminently from the γ-ray collimator region. This is confirmed by the analysis of the distribution of the tracks in the plates exposed without target. For geometric reasons the background is higher on the $\theta = 30°$ plates.

TABLE I.

	H_2O					Background				
Weight (mole)	~ 1					—				
Dose (Röntgen)	3600					2100				
$E_{\gamma\,max}$ (MeV)	31					31				
θ	30°	60°	90°	120°	150°	30°	60°	90°	120°	150°
Scanned volume (mm^3)	183	183	365	177	183	50	95	50	84	88
– Slow scanning – Number of experimental acceptables tracks ($E_n < 5$ MeV)	100	77	190	55	44	12	3	—	2	4
– Slow and fast scanning – Number of experimental acceptables tracks ($E_n > 5$ MeV)	130	174	373	110	72	2	—	—	1	—

We have treated the experimental data taking into account the amount and the distribution of the background tracks.

3·2. *Energy spectra.* – Exposure and scanning data for oxygen target and for background are summarized in Table I.

The experimental neutron spectrum at 90° is shown in Fig. 2*a*. In Fig. 3 are collected the experimental spectra at 30°, 60°, 120° and 150°.

The neutron spectra are inferred from the proton recoil spectra following the method previously adopted ([3]).

Only those tracks, originating and coming to rest in the emulsion, and with a « dip » angle $\beta \leqslant 15°$ and an « azimuth » $\varphi < 30°$ into a pyramid centered in the neutron beam direction, have been considered.

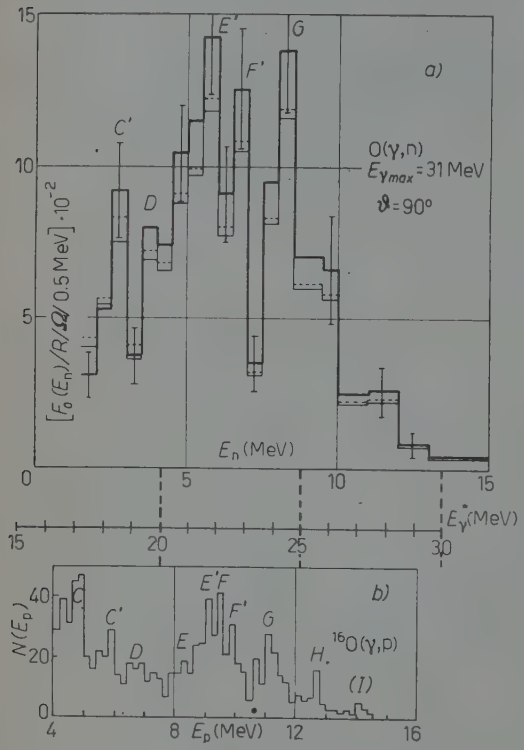


Fig. 2. – *a*) Energy spectrum of the photoneutrons from oxygen at $\theta = 90°$, $E_{\gamma\,max} = 31$ MeV. —— « primary » flux; —— « observed » flux; – – – flux corrected for the absorption in oxygen only. *b*) Energy spectrum of the photoprotons from oxygen at $\theta = 90°$, $E_{\gamma\,max} = 30$ MeV ([13]).

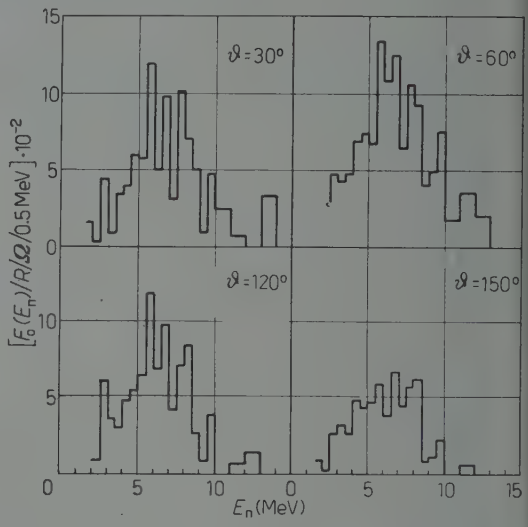


Fig. 3. – Energy spectra of the photoneutrons from oxygen at angles $\theta = 30°$, 60°, 120° and 150°.

The relation $E_n = E_p/\cos^2 \alpha$ (α being the angle between the directions of the recoil proton and the incident neutron) was introduced for $\theta > 10°$.

Corrections due to the probability of escape of the tracks from the 400 μm thick emulsion ([12]) and to the absorption and scattering of neutrons in the water target (Appendix I) give a small modification to the neutron spectrum.

An interpretation of the results is discussed in Section **4**.

3·3. *Angular distribution.* – The angular distributions of the various neutron groups are shown in the following Section.

([12]) L. ROSEN: *Nucleonics*, **11**, 8, 38 (1953).

([13]) C. MILONE, S. MILONE-TAMBURINO, R. RINZIVILLO, A. RUBBINO and C. TRIBUNO: *Nuovo Cimento*, **7**, 729 (1958).

The curves have been fitted on the experimental values by using Legendre polynomials. These distributions will be given and discussed below.

4. – Discussion.

4·1. *Energy spectra.* – The experimental energy spectra at different θ angles. (Fig. 2*a*, 3) agree between themselves and exhibit a fine structure.

It may be noted that the behaviour of the spectra of oxygen is very different from that of all elements previously investigated (see Introduction). For these elements evaporation accounts for the greatest part of the emitted neutrons, whereas in the $^{16}O(\gamma, n)$ spectra the contribution of the direct process is prevailing.

This behaviour so unusual for neutrons is due to the fact that *a*) the neutrons emitted owing to the absorption of γ-rays of energy E_γ between the threshold (15.6 MeV) and about 21 MeV can only leave the residual nucleus in its ground state; *b*) for the γ absorption in the E_γ region $(21 \div 25)$ MeV, the neutrons can reach either the ground state (61%) or the two excited states at 6.14 or at 6.82 MeV (39%) ([14]); *c*) for E_γ higher than 25 MeV the contribution of neutron emission to the total yield is low because the $\sigma_{\gamma, n}$ and the bremsstrahlung spectrum are quickly decreasing. Consequently the contribution of the neutrons which can leave the residual nucleus in excited states is very low.

A remarkable similarity between photoneutrons and photoprotons spectra may be noted ([13]) as shown in Fig. 2-*a*, 2-*b*.

The two spectra refer to θ angles around 90°.

The E_γ^* scale refers to values given by

$$E_\gamma^* = \frac{16}{15} E_p + E_s(\gamma, p) = \frac{16}{15} E_n + E_s(\gamma, n) ,$$

$E_s(\gamma, p) = 12.1$ MeV being the threshold energy of the $^{16}O(\gamma, p)$ process and $E_s(\gamma, n) = 15.6$ MeV the threshold energy of the $^{16}O(\gamma, n)$ process. Therefore E_γ^* is the energy of γ-rays that give rise to the emission of particles of energy E_p or E_n, when the emitted particles leave ^{15}N or ^{15}O respectively in the ground state. Generally is $E_\gamma^* = \frac{16}{15} E_x + E_s(\gamma, x) + E_R$, where E_R is the excited state energy of the residual nucleus.

It may be seen, from the spectra, that in the $^{16}O(\gamma, n)$ process the transitions to excited states of the residual nucleus are less frequent than in the $^{16}O(\gamma, p)$ process. This different situation could be explained simply taking into account

([14]) N. L. SVANTESSON: *Nucl. Phys.*, **3**, 273 (1957).

that $E_s(\gamma, n)$ is 3.5 MeV higher than $E_s(\gamma, p)$, while the maximum of the cross-section is reached for the two processes at about the same E_γ (~ 24 MeV). For this reason in the $^{16}O(\gamma, n)$ reaction the possibility of transitions to excited states is lower than in the $^{16}O(\gamma, p)$ reaction. For E_γ between 20 and 25 MeV the cross-section for ground state transitions, compared to the cross-section for all transitions, are respectively 41% in the $^{16}O(\gamma, p)$ reaction and 61% in the $^{16}O(\gamma, n)$ reaction ([14]).

Finally, we point out that, for energy balance reasons, protons of energy higher than about 8 MeV and neutrons of energy higher than about 6 MeV (Fig. 2, 3) are emitted leaving the residual nucleus in its ground state.

4·2. *Cross-section.* – Measurements on the $^{16}O(\gamma, n)$ cross-section up to 31 MeV ([10]) and 32 MeV ([15]) have been previously made. Both the experiments indicate that the cross-section curves are peaked at about 24 MeV. An half-width $\Gamma = 3.4$ MeV is found by CARVER and LOKAN ([10]) while ERDÖS, SCHERRER and STOLL ([15]) found $\Gamma = 4.2$ MeV.

Now, an analysis of the neutron spectra from oxygen, taking into account the levels of ^{15}O, shows that only a lower part of the spectra—for $E_n < 5$ MeV—may contain a non negligible percentage of neutrons that leave ^{15}O in excited states.

Therefore, taking into account the bremsstrahlung spectrum, it is possible to obtain from the neutron spectra the cross-section for photon absorption giving rise to transitions to the ground state of ^{15}O. This calculation is sufficiently correct for $E_n > 5$ MeV and hence for $E_\gamma > 21$ MeV, while it gives a supervaluation for $E_\gamma < 21$ MeV.

We made the calculations taking into account the irradiation dose and the angular distribution for the various neutron groups by means of the expression:

$$(\sigma^*_{\gamma,n})_{E_{\gamma_i}}^{E_{\gamma_j}} = \left[\frac{4\pi}{\Omega}\,\overline{F}_0(E_n)\Delta E_n\right]_{E_{n_i}}^{E_{n_j}} \Big/ [n_0 I(E_\gamma,\ E_{31})\Delta E_\gamma]_{E_{\gamma_i}}^{E_{\gamma_j}},$$

where: E_n = neutron energy,

$E_{n_j} = E_{n_i} + 0.5$ MeV,

$E_{\gamma_i} = \frac{16}{15} E_{n_i} + E_s(\gamma, n)$,

$\sigma^*_{\gamma,n}$ = cross-section to the ground state,

n_0 = number of oxygen nuclei in the target,

$\frac{\overline{F}_0(E_n)}{\Omega}$ = mean neutron flux per unit solid angle,

$I(E_\gamma, E_{31})$ = bremsstrahlung spectrum with maximum energy $E_\gamma = 31$ MeV ([13]).

([15]) P. ERDÖS, P. SCHERRER and P. STOLL: *Helv. Phys. Acta*, **30**, 639 (1957).

The obtained cross-section is shown in Fig. 4-*b*.

The cross-section measurement is not directly related to our original problem and we think that the obtained values could be affected by small errors. Yet we will discuss them because there is a remarkable agreement between our calculated cross-section to ground state and the results obtained by other writers. So we have reported in Fig. 4-*b* our result (full line) in comparison with the cross-section curves obtained by CARVER and LOKAN [10] and by ERDÖS, SCHERRER and STOLL [15].

Fig. 4-*a* shows Spicer's [16] cross-section for the $^{16}O(\gamma, n)$ reaction from threshold to $E_\gamma = 25.1$ MeV, calculated in 0.5 MeV steps and in 1 MeV steps.

Making a translation of 0.8 MeV in the E_γ scale, a remarkable similarity between our and Spicer's results can be observed.

The coincidence between the peaks E', F' and G in the two different cross sections can be explained by assuming that the peaks correspond prevalently to transitions from resonances in ^{16}O to the ground state. Our peak C' is higher than that observed by SPICER and may contain neutrons emitted in transitions to the 6.14 or 6.82 MeV excited states of ^{15}O.

These results together with the results [10,15] reported in Fig. 4-*b* confirm that the ground state transitions in the $^{16}O(\gamma, n)$ process are prevalent and certainly not lower than the figure of 61% previously found [14].

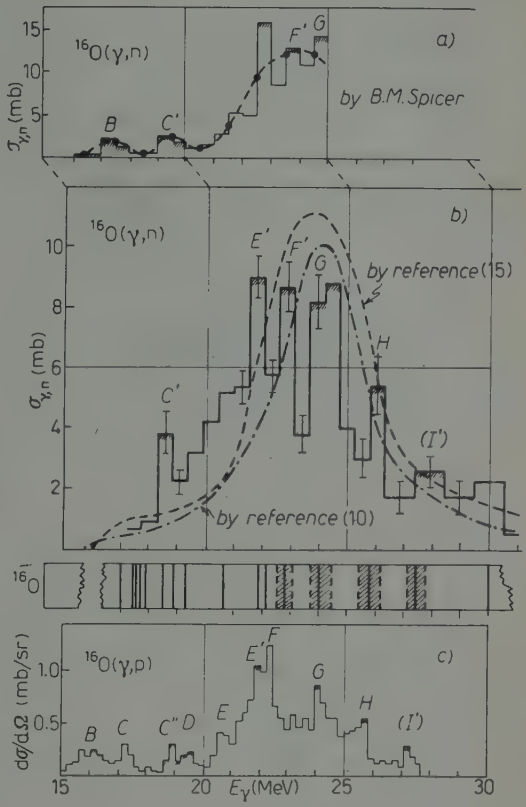


Fig. 4. – *a*) Cross sections for the $^{16}O(\gamma, n)$ reaction, calculated by B. M. SPICER [16] in 0.5 and 1.0 MeV steps. *b*) (γ, n) cross-section to the ground state deduced from our experimental data. – – – $\sigma(\gamma, n)$ given in ref. [15]; –·–·– $\sigma(\gamma, n)$ given in ref. [10]. *c*) Cross section in $O(\gamma, p)$ process to the ground state of ^{15}N [13].

The amount of transitions to excited states in ^{15}O is low and certainly not higher than 40% [14].

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[16] B. M. SPICER: *Austr. Journ. Phys.*, **10**, 326 (1957).

Fig. 4-*c* shows the cross-section to the ground state for the $^{16}O(\gamma, p)$ reaction per unit solid angle at θ around 90° ([13]).

The complete similarity between the $\sigma^*(\gamma, n)$ and the $\sigma^*(\gamma, p)$ section for photoeffect in ^{16}O gives a further confirmation that the peaks observed in the fine structure of the $^{16}O(\gamma, n)$ process are prevalently due to resonances in absorption into the ^{16}O nuclues. That also confirms a many-level theory for the giant resonance of photodisintegration ([16]).

We will assume the neutrons of energy $E_n > 5$ MeV as essentially due to single-particle transitions ([17,18]).

In such hypotheses we think that the coincidence between each individual peak in the (γ, p) and (γ, n) cross-section can be considered as an indication that charge independence of nuclear forces ([19]), when considering ^{16}O single particle excited states up to 30 MeV excitation energy, is a good approximation.

4·3. *Energy levels in the* ^{16}O *nucleus* – The peaks in the $^{16}O(\gamma, p)$ and $^{16}O(\gamma, n)$ spectra for $E^*_\gamma > 21$ MeV (Figs. 2, 3) or those observed in the (γ, p) and (γ, n) cross section to the ground state (Fig. 4) being due to resonances in γ absorption individualize some energy levels in the ^{16}O nucleus.

Our results allow to identify some levels or level groups (see Table II and Fig. 4).

TABLE II.

Peak	Energy of levels in ^{16}O			
	Refer. ([20])	Refer. ([21])	$^{16}O(\gamma, p)$ ([13])	Present work
B	16.0		16 ± 0.3	
C	17.1		17.2 ± 0.2	
C'	18.5			18.5 ± 0.3
C''	18.9		18.9 ± 0.2	
D	19.3		19.4 ± 0.2	
E	20.7		20.7 ± 0.2	
E'	21.9		21.9 ± 0.2	21.8 ± 0.3
F		22.1	22.2 ± 0.2	
F'		23.1		22.8 ± 0.3
G		24.4	24.0 ± 0.2	24.1 ± 0.4
H		25.7	25.7 ± 0.2	26.0 ± 0.3
(*I'*)			27.3 ± 0.2	27.9 ± 0.5

([17]) D. H. WILKINSON: *Physica*, **22**, 1039 (1956).
([18]) A. AGODI: *Nuovo Cimento*, **8**, 515 (1958).
([19]) M. G. MANN and V. L. TELEGDI: *Phys. Rev.*, **91**, 169 (1953).
([20]) F. A. AJZENBERG and T. LAURITSEN: *Rev. Mod. Phys.*, **27**, 77 (1955).
([21]) F. A. AJZENBERG and T. LAURITSEN: *Nucl. Phys.*, **11**, 1 (1959).

In Table II, the levels individualized in our previous work on the $^{16}O(\gamma, p)$ raction ([13]) and in the present work on the $^{16}O(\gamma, n)$ reaction, are compared with the known ^{16}O levels ([20,21]).

In the diagram of ^{16}O nuclear states (Fig. 4), where the level spacing is large, the breadth of the cross hatching indicates the estimated half-width.

4·4. *Angular distribution.* – The experimental results on the $^{16}O(\gamma, n)$ energy spectra compared with the data on the (γ, n) cross-section ([10,14-16]) and with those of the (γ, p) reaction ([13]) strongly suggest that the main contribution to the emission of photoneutrons comes from processes that can be interpreted in terms of one-body excitations ([17,18]) of the target nucleus.

The $E2$ transitions can be neglected as single particle transitions are prevailing. Therefore, the observed asymmetry about 90° implies that there must be interference between $E1$ and $M1$ absorption of photons.

Hence, we use a linear combination of Legendre polynomials, in order to find the theoretical curves that can be fitted on the experimental angular distribution and we will consider only the three first Legendre polynomials P_0, P_1, P_2 ([22,23]). Namely we will analyse the angular distributions by means of functions of the type

$$I(\theta) = a + b \cos\theta + c\left(\frac{3\cos^2\theta - 1}{2}\right).$$

In the photoneutron angular distributions we have analysed separately the neutron groups as a function of their energy.

I) *Neutron group with* $(9 < E_n < 10)$ MeV. The angular distribution of this group corresponding to the peak H is shown in Fig. 5-*a*. These neutrons, due to photons of energy $E_\gamma > E_{\sigma\,max}$, leave the ^{15}O in the ground state.

The values of the *a*, *b*, *c* coefficients are reported in Table III.

TABLE III. – *Values of a, b, c for the angular distributions.*

Neutron energy (MeV)	Peak	Fig.	*a*	*b*	*c*
9 ÷10	*H*	5 *a*	0.62	0.12	— 0.45
5 ÷ 9	*G*, *F'*, *E'*	5 *b*, *c*, *d*, *e*	0.80	0.20	— 0.30
3.5÷ 5		5 *f*	0.73	0.07	— 0.45
2 ÷ 3.5	*C'*	5 *g*	0.65	—	— 0.50

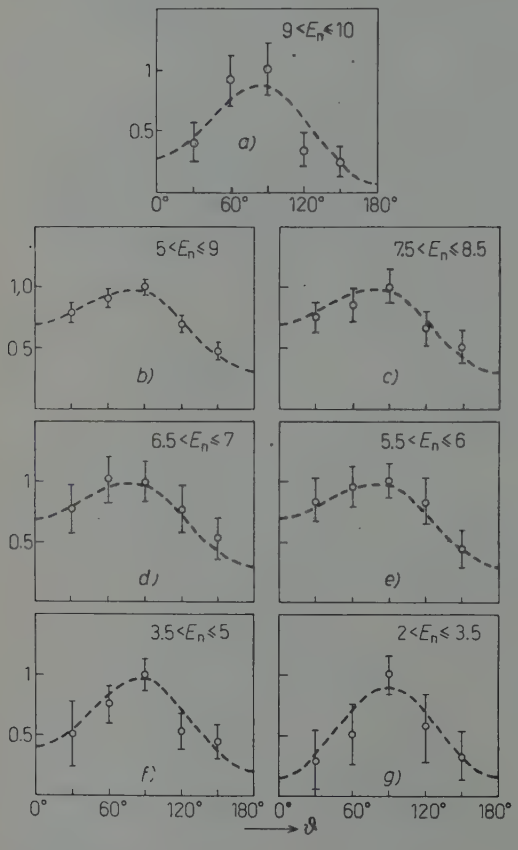
([22]) C. N. YANG: *Phys. Rev.*, **74**, 764 (1948).
([23]) J. M. BLATT and L. C. BIEDENHARN: *Rev. Mod. Phys.*, **24**, 258 (1952).

II) $(5 < E_n < 9)$ MeV. – These neutrons are in the main due to γ-absorption in the region of the giant resonance and are emitted in transitions to the ground state of ^{15}O.

This group contains the three peaks G, F', E'. Both the whole group and the three peaks may be fitted by the same theoretical curve: see Fig. 5-*b*,*c*,*d*,*e*. The values of the coefficients a, b, c are given in Table III.

III) $(3.5 < E_n < 5)$ MeV. – These neutrons are due to transitions to various levels in the residual nucleus.

The back and front asymmetry is smaller than that observed in the previous groups as Fig. 5-*f* and the a, b, c values reported in Table III show. That implies that only a type of transition is prevailing in the various processes involved.



IV) $(2 < E_n < 3.5)$ MeV. – In this region our results are affected by large errors because of background tracks and there could be a mixture of neutrons emitted owing to absorption of photons of various E_γ.

These neutrons corresponding to the peak C' seem to have an angular distribution with symmetry around 90°. That could correspond to pure transition in the processes involved.

For neutrons of energy higher than 10 MeV, due to photons of energy $E_\gamma > 26$ MeV the angular distributions are not given as the experimental data are poor.

Fig. 5. – Angular distributions of photoneutrons from oxygen for the various neutron groups.

5. – Conclusion.

1) For the $^{16}O(\gamma, n)$ reaction, the contribution of the direct process is prevalent, while for all elements investigated up to this date thee vaporation accounts for the greater part of the emitted neutrons.

The transitions to the ground state amount to at least 60% of all neutrons emitted from ^{16}O.

2) The neutron spectra show a fine structure in good agreement with the proton spectra and the (γ, n) and (γ, p) cross-sections.

The individualized levels in ^{16}O up to 28 MeV confirm many previously known levels. Some levels known with uncertainty are also confirmed.

3) The angular distributions, analysed separately for single neutron groups show back and front asymmetry in the $5 < E_n < 10$ MeV region.

4) Experimental results suggest that:

a) a many-level theory for the giant resonance of ^{16}O photodisintegration is indicated;

b) the processes involved in the greatest part of photoneutron emission from ^{16}O can be interpreted in terms of one-body transitions;

c) in these hypotheses, the charge independence of nuclear forces up to 30 MeV excitation energy is a good approximation.

* * *

We thank Prof. R. Ricamo and Prof. G. Wataghin who put at our disposal the means to accomplish the present work.

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Appendix

Calculation of neutron energy spectra deformation for elastic and non-elastic collisions in H_2O.

A.1. *Approximation to the case of a point source.* – The attenuation of the γ-beam through the H_2O target is very low in our experiment, and therefore we take the target as a sphere which contains a uniform distribution of neutron sources.

It may be shown that the mean range $\sqrt{\bar{x}^2}$ of the neutrons inside the sphere of radius R, is about 0.9 R.

Being (Fig. 6)

$$x_r = r\cos\theta + \sqrt{R^2 - r^2\sin^2\theta}\,,$$

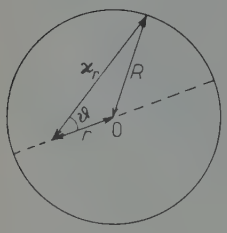

Fig. 6.

the mean square range made into the target by neutrons originated at a distance r from the center of the sphere is given by

$$\bar{x}_r^2 = \frac{\int x_r^2\,\mathrm{d}\Omega}{\int \mathrm{d}\Omega} = \frac{1}{2}\int_0^{\pi} x_r^2 \sin\theta\,\mathrm{d}\theta = R^2 - \frac{r^2}{3}\,.$$

Taking into account the « weight » of the various elementary volumes $\mathrm{d}V_r = 4\pi r^2\,\mathrm{d}r$ on this mean value, we have

$$\bar{x}^2 = \frac{\int \bar{x}_r^2\,\mathrm{d}V_r}{\int \mathrm{d}V_r} = \frac{4\pi\int_0^R \bar{x}_r^2 r^2\,\mathrm{d}r}{\frac{4}{3}\pi R^3} = \frac{4}{5}R^2\,,$$

and hence

$$\bar{x} = \sqrt{\bar{x}^2} = \sqrt{\tfrac{4}{5}}\,R \approx 0.9R\,.$$

For neutrons with energy of some MeV or higher, the range in the target is very low in comparison with the scattering mean free path λ_{sc}; so with good approximation for our calculations, we may suppose the probability of the neutron interaction proportional to the range x for a determined energy of neutrons.

Namely we have

$$\overline{p^*(E)} = \overline{1 - \exp\left[-(x/\lambda^*)\right]} \approx \left(\overline{\frac{x}{\lambda^*}}\right) = \bar{x}/\lambda^*\,,$$

$p^*(E)$ being the mean probability of a particular interaction in the range x, and $\bar{x}/\lambda^*$ being small with respect to unity.

Then in the following paragraphs, the uniform distribution of neutron sources, of strength S per unit volume in the sphere of radius R, will be substituted with a point source in the center of the sphere and having intensity $S' = S\cdot\frac{4}{3}\pi\bar{x}^3$, where $\bar{x} \simeq 0.9R$.

A.2. *Assumptions which allow the development of the calculations of paragraph* A.3. – 1) The scattering mean free path of neutrons for collision with the hydrogen of the target is large with respect to the radius of the target. In fact $\bar{x} \simeq 1.5$ cm while $\lambda_{sc,\,H} = (5.2, 9.2, 15.9, 23.2)$ cm for neutrons of energy (2, 5, 10, 15) MeV respectively.

Then in the calculations we will not take into account the multiple scattering.

2) In the energy region between 2 and 15 MeV the neutron total cross-section for Oxygen may be attributed in about equal measure to elastic as well as to nonelastic collisions. Experimental measurements of the neutron nonelastic cross-sections in the $(7 \div 14)$ MeV range ([24]) on several elements show that the elastic and nonelastic cross-sections are not considerably different.

Hence in the following paragraph we will make the assumption that for the neutron cross-section on Oxygen, $\sigma_{te} \approx (\sigma_T/2)$.

3) In the energy region of several MeV, the collision with hydrogen can be considered as elastic and the binding energy between atoms quite negligible.

The energy distribution of the neutron after an elastic collision with a nucleus of mass A is given by

$$\frac{\mathrm{d}N(E_n)}{\mathrm{d}E_n} = \frac{(A+1)^2}{A}\,\frac{\pi}{E_0}\,\sigma(\beta, \psi_c)\,,$$

$\sigma(\beta, \psi_c)$ being the scattering differential cross-section in the c.m. system. As the scattering is isotropic in the c.m. system for the neutron energy until several MeV, it is

$$\frac{\mathrm{d}N(E_n)}{\mathrm{d}E_n} = \frac{(A+1)^2}{4AE_0} = \text{const}\,.$$

The maximum and minimum neutron energies in the laboratory system after a collision are E_0 and $E_0[(A-1)^2/(A+1)^2] = E_0\alpha$, respectively, where E_0 is the initial neutron energy; all final energies from E_0 to αE_0 are equally probable.

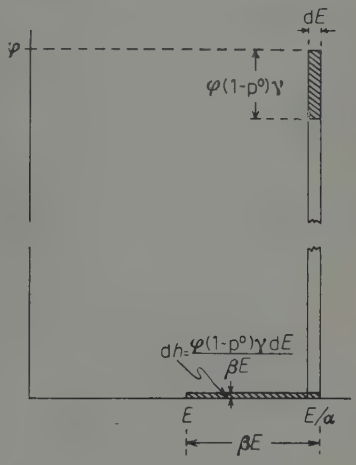

Fig. 7.

Hence in the case of a neutron of initial energy E/α the probability that its energy lies between an interval $\mathrm{d}E$ between E and E/α is given by

$$\frac{\mathrm{d}E}{(E/\alpha) - E} = \frac{\mathrm{d}E}{E((1/\alpha) - 1)} = \frac{\mathrm{d}E}{\beta E} \qquad \text{(see Fig. 7)}.$$

We have $\alpha = 0$, $\beta = 1$ for hydrogen and $\alpha = 0.78$, $\beta = 0.28$ for oxygen.

A.3. *Calculation of the spectra deformation.* – Let be:

f	the « observed » flux of neutrons;
φ	the « primary » flux of neutrons;
x	the mean distance travelled by neutrons;
$p = \exp[-\sigma_H n_H x]$	the probability for a neutron not to collide with hydrogen;
$p^0 = \exp[-\sigma_O n_O x]$	the probability for a neutron not to collide with oxygen;
$1 - p$	the probability of interaction in hydrogen;
$1 - p^0$	the probability of interaction in oxygen;

([24]) W. P. Ball, M. MacGregor and R. Booth: *Phys. Rev.*, **110**, 1392 (1958).

we make the positions

$$P = p + p^0 - 1 ,$$

and (A.2, 2)

$$\left(\frac{\sigma_{sc}}{\sigma_T}\right) = \gamma \approx \frac{1}{2} \text{ for oxygen .}$$

Therefore, for a given « observed » energy E_k, taking into account all possible interactions with hydrogen as well as with oxygen, one obtains that the « observed » flux is correlated to the « primary » flux by the following integral equation:

$$(1) \quad f_k = (1-(1-p_k)-(1-p_k^0))\,\varphi_k + \int_{E_k}^{\infty} \varphi(1-p)\,\frac{dE}{E} + \int_{E_k}^{E_k/\alpha} \varphi(1-p^0)\gamma\,\frac{dE}{\beta E} =$$

$$= P_k\varphi_k + \int_{E_k}^{\infty} \varphi(1-p)\,\frac{dE}{E} + \frac{\gamma}{\beta}\int_{E_k}^{E_k/\alpha} \varphi(1-p^0)\,\frac{dE}{E} .$$

where the terms to state the contributions to the « observed » flux are respectively given by:

i) neutrons of energy E_k which had collisions in the target;

ii) neutrons of energy E_k which had no collisions in the target;

iii) neutrons of energy higher than E_k which have had collisions in the hydrogen of the target;

iv) neutrons of energy between E_k and E_k/α which have had collisions in the oxygen of the target.

Now, in order to obtain a relation connecting the « primary » to the « observed » flux, we differentiate the (1) with respect to the energy E in the point K:

$$\left(\frac{df}{dE}\right)_k = \left(\frac{dP}{dE}\right)_k \varphi_k + P_k\varphi_k' - \frac{\varphi_k(1-p)}{E_k} + \frac{\gamma}{\beta}\left[\frac{1}{\alpha}\varphi\left(\frac{E_k}{\alpha}\right)\left[1-p^0\left(\frac{E_k}{\alpha}\right)\right]\frac{\alpha}{E_k} - \right.$$

$$\left. - \varphi_k(1-p_k^0)\,\frac{1}{E_k}\right] .$$

Observing that

$$\varphi\left(\frac{E_k}{\alpha}\right) = \varphi_k + \varphi_k'\left(\frac{E_k}{\alpha} - E_k\right) = \varphi_k + \beta E_k\varphi_k' ,$$

$$p^0\left(\frac{E_k}{\alpha}\right) = p_k^0 + p_k^{0\prime}\left(\frac{E_k}{\alpha} - E_k\right) = p_k^0 + \beta E_k p_k^{0\prime} ,$$

one obtains:

$$f' = P'\varphi + P\varphi' - \frac{1-p}{E}\,\varphi + \frac{\gamma}{\beta}\left[(\varphi+\beta E\varphi')(1-p^0-\beta E p^{0\prime})\,\frac{1}{E} - \varphi(1-p^0)\,\frac{1}{E}\right] =$$

$$= P'\varphi + P\varphi' - \frac{1-p}{E}\,\varphi + \gamma(1-p^0-\beta E p^{0\prime})\varphi' - \gamma p^{0\prime}\varphi ,$$

from which

$$\varphi' + \frac{P' - \gamma p^{0\prime} - (1-p)/E}{P + \gamma(1 - p^0 - \beta E p^{0\prime})}\,\varphi = \frac{f'}{P + \gamma(1 - p^0 - \beta E p^{0\prime})}\,.$$

Let

$$a(E) = \frac{P' - \gamma p^{0\prime} - (1-p)/E}{P + \gamma(1 - p^0 - \beta E p^{0\prime})}; \qquad b(E) = \frac{f'}{P + \gamma(1 - p^0 - \beta E p^{0\prime})},$$

we have

$$\varphi' + a(E)\varphi = b(E)\,. \tag{2}$$

This equation is solved by the following function φ, from which the « primary » flux at the energy E_i can be obtained:

$$\varphi(E_i) = \exp\left[-\int_0^{E_i} a(E)\,\mathrm{d}E\right]\left[\text{Const.} - \int_0^{E_i} b(E)\,\exp\left[\int_0^{E_i} a(E)\,\mathrm{d}E\right]\mathrm{d}E\right]. \tag{3}$$

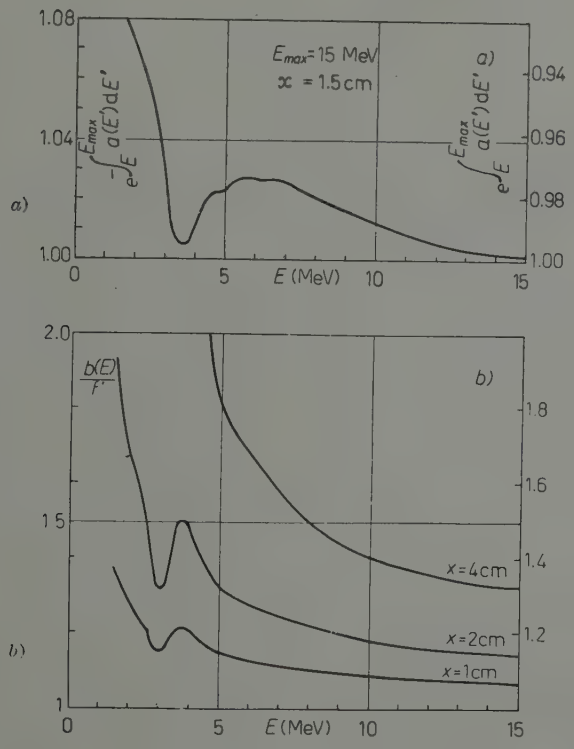

Fig. 8.

Now it is preferable to integrate the eq. (3) between E and the maximum energy of the spectra rather than between zero and E.

Therefore indicating with E the values of energy between E_i and E_{max}, and indicating with E' the values of energy between E and E_{max}, by means of eq. (3) one finds that the « primary » flux at the neutron energy E_i is given by:

$$\varphi(E_i) = \exp\left[\int_{E_i}^{E_{max}} a(E')\,dE'\right]\left[A^2 - B^2\int_{E_i}^{E_{max}} b(E)\exp\left[-\int_{E}^{E_{max}} a(E')\,dE'\right]dE\right].$$

The Fig. 8*a* shows the values of $\exp\left[\int_{E}^{E_{max}} a(E')\,dE'\right]$ and of $\exp\left[-\int_{E}^{E_{max}} a(E')\,dE'\right]$ as a function of $E_n = E$ for $x = 1.5$ cm of H_2O target and for $E_{n\,max} = 15$ MeV.

The Fig. 8*b* shows the values of $b(E)/f'$ as a function of E $(=E_n)$ for $x = 1$, 2 and 4 cm of H_2O respectively. The values of the constants A^2 and B^2 are determined from the experimental values of the neutron spectra in the highest energy region.

RIASSUNTO

Si studiano gli spettri energetici e le distribuzioni angolari dei fotoneutroni emessi dall'ossigeno irradiato con bremsstrahlung di 31 MeV. Si è adoperata una targhetta di acqua e si è calcolata la piccola deformazione degli spettri neutronici in seguito a collisioni dei neutroni nella targhetta stessa. I fotoneutroni emessi ad angoli $\theta = 30°$, 60°, 90°, 120° e 150° vengono rivelati utilizzando i protoni di rinculo osservati in emulsioni nucleari Ilford L-4 di 400 μm di spessore. Il fondo è risultato trascurabile per energie dei neutroni maggiori di 5 MeV. Soltanto per energie inferiori a 3 MeV si ha un notevole numero di tracce spurie particolarmente per $\theta = 30°$ e 150°. Gli spettri dei neutroni ottenuti ai vari angoli ed analizzati ad intervalli di 0.5 MeV presentano una stessa struttura fine che concorda ottimamente con quelle osservate negli spettri dei fotoprotoni e nelle sezioni d'urto $O(\gamma, n)$ ed $O(\gamma, p)$. Si individuano parecchi livelli dell'^{16}O che concordano con i livelli già noti e si confermano altri livelli incerti. Nell'emissione di neutroni il processo diretto risulta preponderante e le transizioni verso lo stato fondamentale del ^{15}O sono almeno dell'ordine del 60%. Le distribuzioni angolari vengono analizzate separatamente per ciascun gruppo di neutroni e nella regione della risonanza gigante si osserva un'asimmetria attorno a 90°. I risultati ottenuti possono, con buona approssimazione, essere interpretati in termini di: *a*) teoria a più livelli per la risonanza gigante; *b*) transizioni di particella singola; *c*) indipendenza dalla carica delle forze nucleari.

On the 1409 keV Transition in ^{152}Sm.

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Tata Institute of Fundamental Research - Bombay

(ricevuto il 3 Giugno 1959)

In the studies made in this laboratory on the decay of enriched ^{152}Eu the 1409 keV γ-ray was reported by Bhattacherjee *et al.* [1] to be not in coincidence with the 121.8 keV γ-ray in the electron capture branch. The same two γ-rays were, however, found by other workers [2-5] to be in coincidence. The present experiment was, therefore, undertaken to clarify this main disagreement.

A usual fast-slow coincidence scintillation spectrometer incorporating a ten channel pulse height analyser [6] recently built in the laboratory was employed. The fast coincidence circuit has a resolving time $2\tau = 16$ μs. The chance coincidences were thus negligible. This fact was also checked by delaying one of the inputs to the fast coincidence circuit by introducing a long delay cable. The experiment was performed with the two counters at 90°. The results are shown in Fig. 1*a* and *b*. Fig. 1*a* shows the coincidences in the high energy region with the 121.8 keV γ-ray whereas Fig. 1*b* shows the coincidences with the 1409 keV γ-ray. These two experiments clearly show that the 1409 keV γ-ray is fully in coincidence with the 121.8 keV γ-ray giving the highest energy level in ^{152}Sm at 1531 keV. The discrepancy between the results of our earlier work and those of others is thus now definitely known to be experimental in origin.

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(1) S. K. Bhattacherjee, T. D. Nainan, S. Raman and B. Sahai: *Nuovo Cimento*, **7**, 501 (1958).

(2) L. Grodzins: *Bull. Am. Phys. Soc.*, **1**, 163 (1956).

(3) O. Nathan and M. A. Waggoner: *Nucl. Phys.*, **2**, 548 (1957); O. Nathan and S. Hultberg: *Nucl. Phys.*, **10**, 118 (1959).

(4) J. M. Cork, M. K. Brice, R. G. Helmer and D. E. Sarason: *Phys. Rev.*, **107**, 1621 (1957).

(5) S. Ofer: *Nucl. Phys.*, **4**, 477 (1957).

(6) C. W. Johnstone: *Nucleonics*, **11**, 36 (1953).

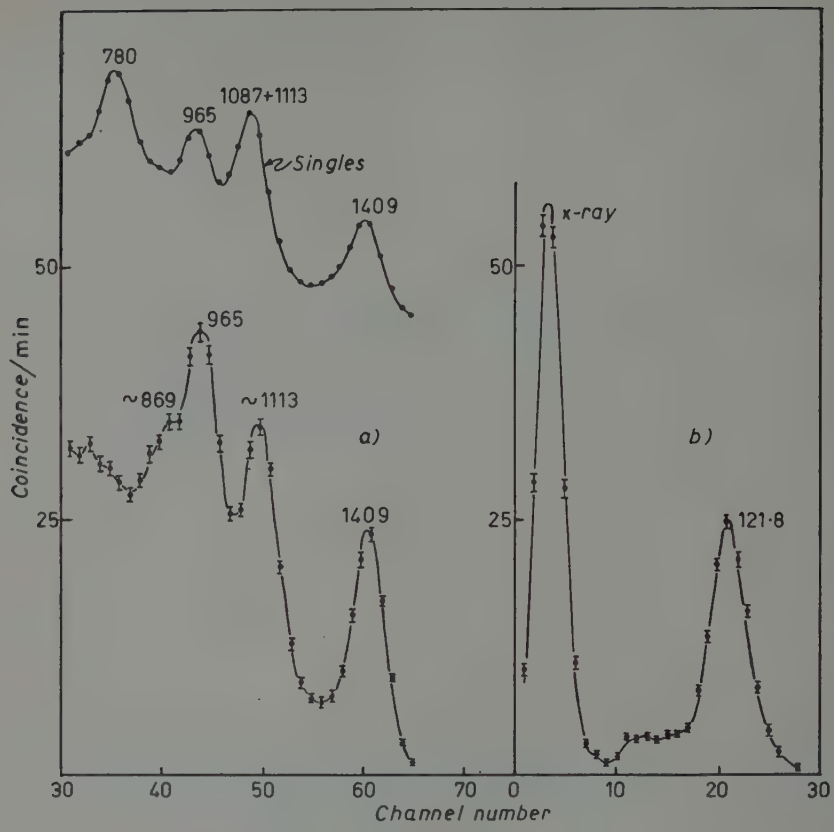

Fig. 1. – *a*) High energy γ-spectrum in coincidence with the 121.8 keV γ-ray in the electron capture decay of ^{152}Eu. Singles' spectrum is shown for comparison. Energies are in keV. *b*) X-rays and the 121.8 keV γ-ray in coincidence with the 1409 keV γ-ray.

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The Cosmic Ray Storm of May 11, 1959.

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Commissione Italiana Anno Geofisico Internazionale

(ricevuto l'11 Giugno 1959)

A marked sudden decrease of the Cosmic Ray (C.R.) intensity was observed in Rome on the night of May 11, 1959 in connection with a geomagnetic storm with sudden commencement (S.C.).

This is the most outstanding Forbush-type event which has been detected at this C.R. station since the beginning of the I.G.Y.

Figs. 1 and 2 show the response of the three C.R. detectors, which are permanently operated at the University of Rome (41° 54′ N, 12° 31′ E geogr.; 42.5° geom. conv.; s.l.); they are:

1) A neutron monitor of standard I.G.Y. type, consisting of 12 BF_3 counters, to record the « Nucleonic Component ».

2) A threefold coincidence apparatus of G.M. counters, to record:

a) the « Meson Component » (10 cm Pb);

b) the « Total Ionizing Component » (unshielded telescopes having two trays in common with the telescopes of point *a*)).

All the data are corrected for pressure effect. The normal mean rates are as follows:

— Nucleonic Component: 15.730 counts/h;

— Total Ionizing Component: 49.630 counts/h;

— Meson Component: 72.960 counts/h.

In Fig. 1 the C.R. data are plotted with a resolving time of half an hour, to stress the sharp slope of the intensity decrease.

In Fig. 2 bihourly values of the C.R. intensities are presented for several days after May 11, in order to illustrate the slow rise toward the pre-storm level; this was not yet reached even about 20 days later.

It might be interesting at this point to notice, overimposed on the long recovery phase, the occurrence of two further, much smaller events: *i.e.* those of May 13-14 and May 24, the second of them being also associated with a geomagnetic storm (Sudden Commence-

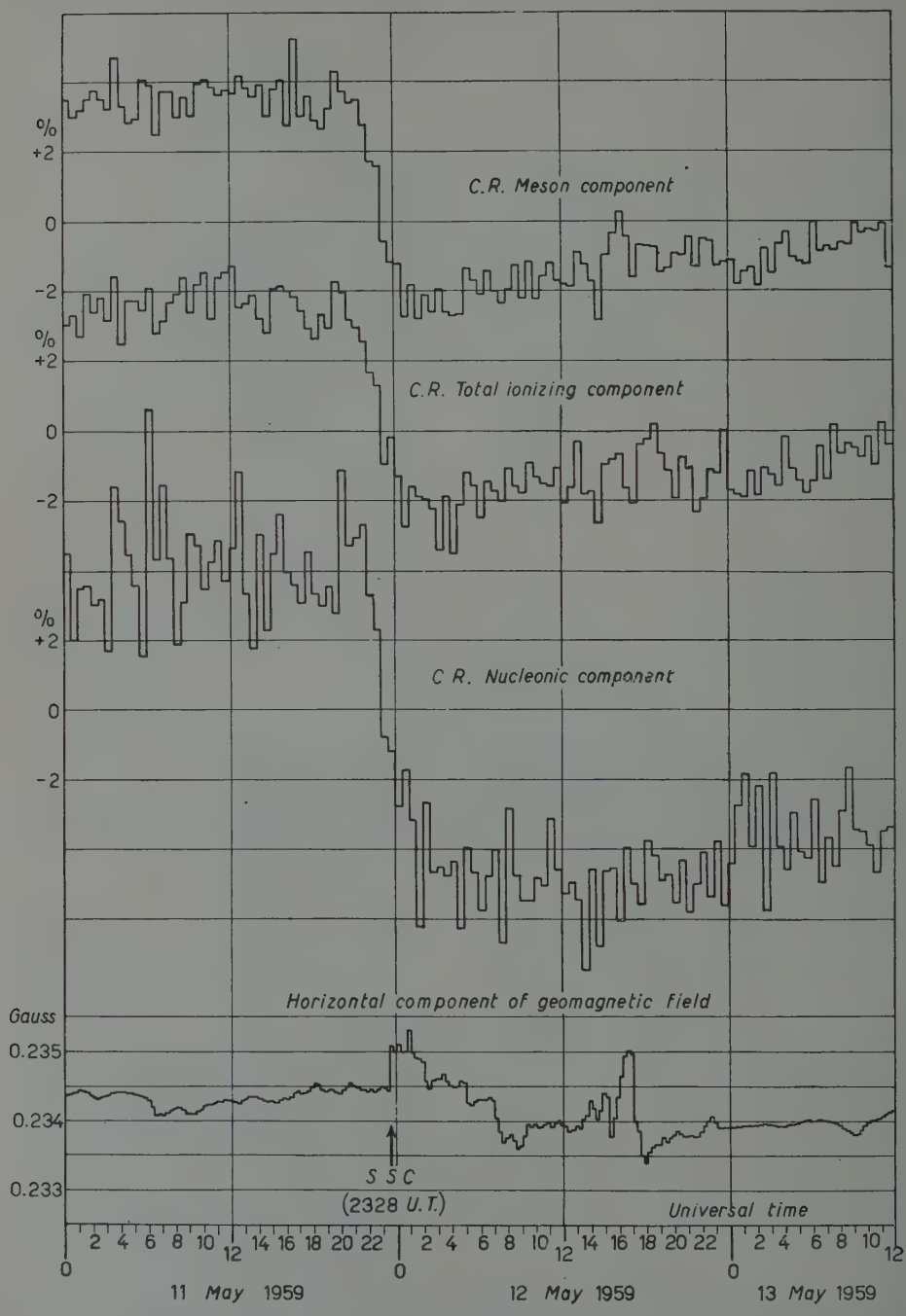

Fig. 1. — 30-min mean values of the pressure corrected C.R. Components (Standard Errors: Meson 0.5 %; Total 0.6 %; Nucleonic 1.1 %). 15-min mean values of the Horizontal Component of the geomagnetic field.

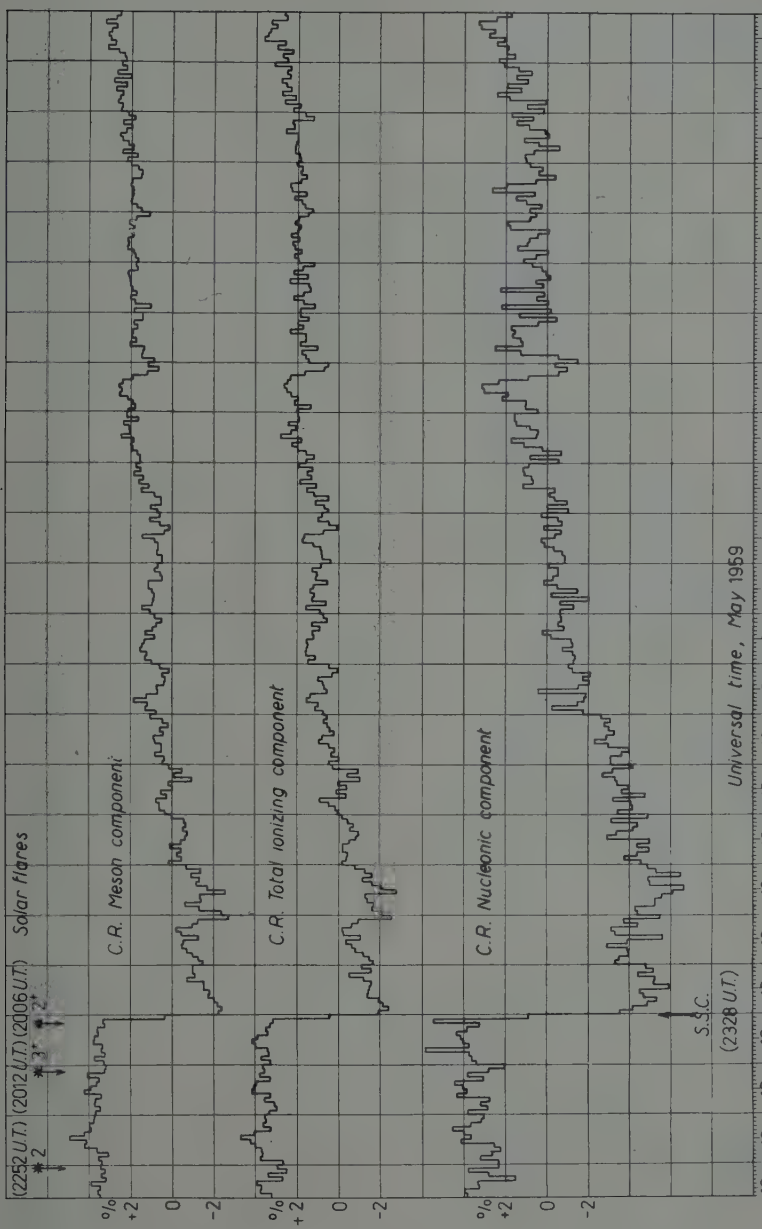

Fig. 2. – Bi-hourly values of the pressure corrected C.R. Components (Standard Errors: Meson 0.3 %, Total 0.3 %, Nucleonic 0.6 %).

ment of 8 γ, as detected at the Geomagnetic Observatory of L'Aquila, on May 24, 1959, 0540 U. T.). The world-wide character of these events might be established only when the data of other C.R. stations will be available.

The main parameters computed for the C.R. event of May 11 are given in the following Table:

TABLE I.

C.R. component	Starting time U.T.	Amplitude of the decrease (%)	Slope of the decrease (% per h)
Nucleonic	22 00 $\pm$ 30	9.0 $\pm$ 0.3	$\sim$ 4
Total ionizing		5.7 $\pm$ 0.2	$\sim$ 2.5
Meson		5.7 $\pm$ 0.2	$\sim$ 2.5

The geomagnetic storm which occurred in association with the main Forbush-Decrease is also showed in Fig. 1, as recorded at the Observatory of L'Aquila (42° 23′ N, 13° 19′ E geogr.; 42.9° N geom.) [1]. The 15-min mean values of the horizontal component of the earth magnetic field are there reported. In this storm a very large S.C. (69 γ) at 23 28 U.T. was followed by an initial phase of 122 γ maximum amplitude, and then, through a quite slow fall, by a main phase comparatively not very intense: the total range was of about 240 γ. The same general characteristics and the same S.C. time are apparent in the recording of the Gibilmanna (37° 59′ N, 14° 01′ E geogr.; 38° 28′ N geom.) and the Asiago (45° 52′ N, 11° 31′ E geogr.; 46° 36′ N geom.) Observatories.

It may be worthwhile to mention that the geomagnetic planetary indices (K_p) of May 12, as reported by Dr. J. BARTELS, Göttingen, are quite high during the whole day; this confirms the world-wide character of the magnetic storm. The three-hourly values of the K_p-indices and the corresponding local indices, as supplied by L'Aquila (K_{Aq}) are listed in Table II for May 11, 12 and 13.

Also the solar activity was very high in the period under study. In Table III a list of the outstanding solar flares occurred during the immediately preceding days is reported, as communicated by Dr. D. TROTTER, High Altitude Observatory, Boulder, Colorado:

TABLE II.

Index								
	May 11							
K_p	3_0	$2+$	4_0	3_0	$4-$	$4-$	$4-$	6_0
K_{Aq}	3	3	3	2	3	3	3	6
	May 12							
K_p	7_0	$7-$	$7-$	$6-$	$7-$	$8+$	$5+$	$5-$
K_{Aq}	6	5	5	4	5	7	4	4
	May 13							
K_p	$2-$	$2+$	$4-$	$3-$	$3-$	3_0	3_0	$3+$
K_{Aq}	2	2	3	3	3	2	3	4

If any association between a solar flare and both geomagnetic and C.R. storms is to be looked for, the flare most likely associated in the present case would seem that of importance 3+

[1] Private communication from Dr. F. MOLINA, of the Istituto Nazionale di Geofisica, Roma.

TABLE III.

Day	Optical Flare, U.T.	Importance	Radio Burst, Frequency, MHz
May 8	22 52 ÷ 23 22	2	Major, 10000, 545, 200, 167
May 10	20 12 ÷ 23 42	3+	200, 167
May 11	20 06 ÷ 21 45	2+	545, 200, 167

occurred on May 10 (about 26 hours in advance). This is also suggested by a systematic examination of several events similar to that of May 11, as carried our for a whole period of 13 months ([2,3]).

The main features of this C.R. storm are in common with the most interesting events occurred during this solar maximum (see ([2])): *i.e.* essentially the large and sharp drop in the intensity, the comparatively slow recovery, the association with a S.C. geomagnetic storm and with a large solar flare, in advance of about one day and accompanied by a radio burst.

Particularly relevant in this case are, however, the amplitude and the slope of the decrease. Moreover, an interesting feature of this event may be the fact that the S.C. of the geomagnetic storm occurs when the C.R. intensity has unmistakably fallen well below the initial value. Such an occurrence seems a rather unusual one, at least according to what the results of the above mentioned investigation suggest. This view was mainly derived from the inspection of the C.R. event as detected in Rome, with a 15-min resolving time. But it may be confirmed only by the comparison with the results of other stations, since a time shift of the initial phase of such C.R. events with longitude was noticed in some cases, in spite of the poor resolving times generally provided (bi-hourly data).

These circumstances altogether might imply an interesting aspect of the phenomena here concerned, although the inadequacy of elements so far available cannot provide any clear indication as yet. We believe that a more thorough and careful investigation on this line should be worthwhile, in order that a better understanding be gained of the mechanism by which solar perturbations trigger both C.R. and geomagnetic storms.

* * *

We are deeply indebted to Dr. F. MOLINA both for the geomagnetic data and for useful discussions. We are grateful to Miss D. TROTTER for supplying solar data in advance.

([2]) F. BACHELET, P. BALATA, A. M. CONFORTO and G. MARINI: *The Nucleonic Component of C.R. Intensity in Rome, from July 1, 1957 to July 31, 1958*, to be published.

([3]) *Note added in proof.* – More details about this flare have been provided by the Preliminary Report of Solar Activity, TR 402 of May 15, 1959, of H.A.O., Boulder, Colorado. In particular it has been reported that this flare was the largest observed in this solar cycle and was also linked with: 1) a strong aurora, on May 11; 2) a very strong polar cosmic noise absorption; 3) a high altitude cosmic ray intensity of fifty times normal.

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Beam Maser Spectroscopy on HDO (*).

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(ricevuto il 30 Luglio 1959)

A beam maser of the type originally used by Gordon, Zeiger, and Townes [1,2] to investigate the magnetic hyperfine structure of the NH_3 inversion lines has been used to resolve the hyperfine structure of the $2_{20} - 2_{21}$ rotational transition in HDO which lies at 10 278.245 MHz. The quadrupole satellites reported by Posener [3] have been fully resolved, and substantial effects due to magnetic interactions confirmed [4].

HDO is one of the most favorable molecules to use in a beam maser spectrometer. Its small moments of inertia, comparable to those of NH_3, give a rotational spectrum which in general lies in the far infrared, and consequently a much heavier population of the lower rotational states than is the case for molecules whose characteristic rotational transitions lie in the microwave region. The departure of this asymmetric rotor from a prolate symmetric top ($K = -0.6841$) slightly splits the $J = 2$, $K = 2$ level into the 2_{20} and 2_{21} levels which, due to their relative isolation from the other rotational states, experience a Stark effect very favorable to the electrostatic focusing used in a beam maser.

The beam maser used is similar in its essentials to one previously constructed in this Laboratory [1]. The reflection cavity operated in the TM_{010} mode with a Q of 11 000. It was machined to a resonant frequency approximately 6 MHz higher than the expected frequency of the HDO transition, and thermally tuned to the line itself.

The low saturation level of the emission lines, of the order of 10^{-10} W, demands highly sensitive detection techniques. A superheterodyne receiver requiring only a single klystron was obtained by mixing the klystron output at a crystal mixer with 30 MHz from a crystal-stabilized oscillator and using one of the side bands as the signal frequency. Sufficient stability of the klystron, a Varian X-13, for the very

(*) Work supported jointly by the U. S. Army Signal Corps, the Office of Naval Research and the Air Force Office of Scientific Research.

(**) National Science Foundation Postdoctoral Fellow. Permanent address: Physics Department, University of the Orange Free State, Bloemfontein.

(1) J. P. Gordon, H. J. Zeiger and C. H. Townes: *Phys. Rev.*, **95**, 282 (1954); **99**, 1264 (1955).

(2) J. P. Gordon: *Phys. Rev.*, **99**, 1253 (1955).

(3) D. W. Posener: *Austr. Journ. Phys.*, **10**, 276 (1957).

(4) D. W. Posener: private communication.

narrow emission lines observed, was obtained by careful mechanical and thermal isolation. The repeller and filament voltages were furnished by batteries.

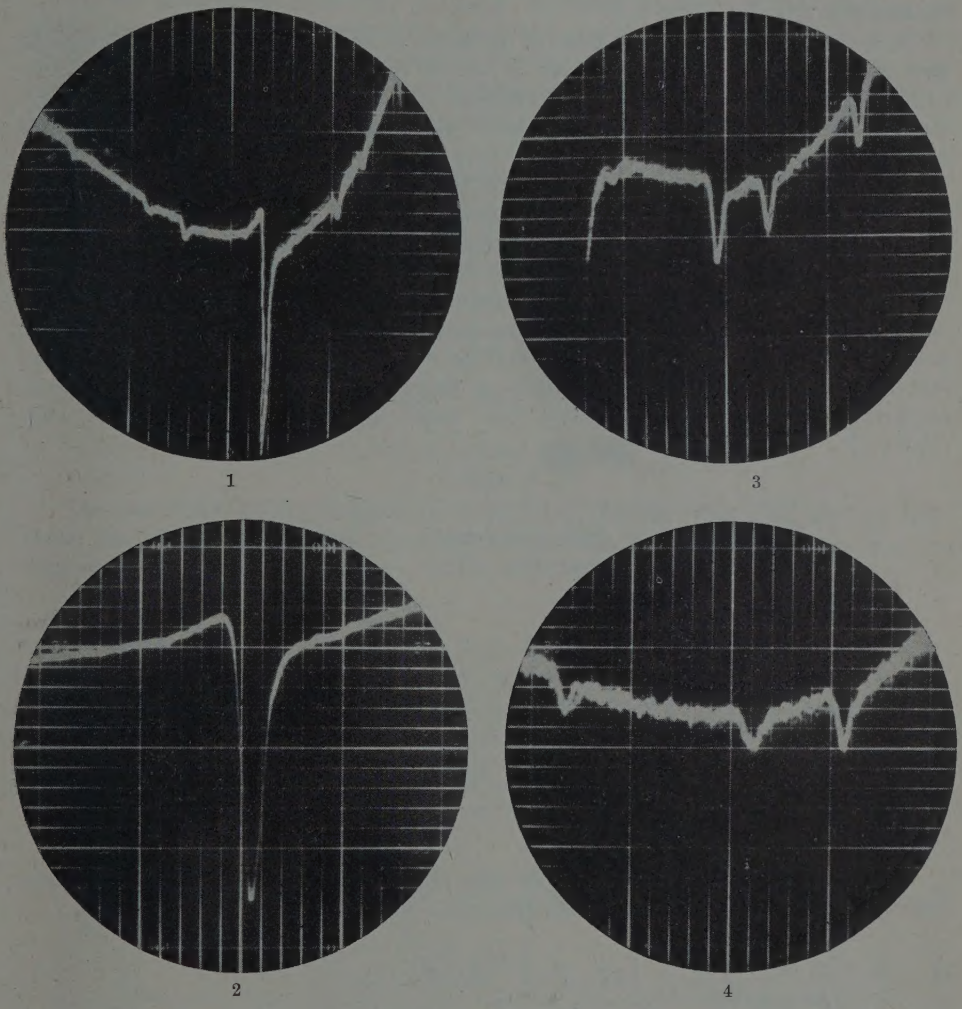

Fig. 1. – Oscilloscope traces of the hyperfine structure of the HDO 2_{20}-2_{21} transition. Frequency increases to the right. The hyperfine lines are labeled A, B, C, D, proceeding *away* in either direction from the central transition. The upper left photograph (1) shows the full spectrum against the background of the cavity response; the lower left (2) shows the main $\Delta F_1 = 0$ transition itself. The upper (3) and lower (4) traces on the right are of the high and low frequency components respectively. The weak $1\frac{3}{2}$-$3\frac{5}{2}$ « A » transitions can barely be made out close to the main line in the first two photographs; the high frequency one is seen clearly in the third picture, close to the left hand edge of the trace. It is the only one of the two which we have succeeded in measuring by the technique described.

The hyperfine structure of the $2_{20} - 2_{21}$ transition is shown in Fig. 1. The apparent relative weakness of the low frequency hyperfine components compared to their high frequency counterparts is a real effect, caused by preferential focusing

of the higher hyperfine levels in the 2_{20} state. A similar effect has been observed in the quadrupole satellites of the NH_3 inversion lines using a beam maser ([2]). The measured line widths of 5 kHz approach closely the theoretical limit imposed by the uncertainty relation $\Delta\nu\,\Delta t \approx 1$.

It is calculated from the observed strength of the central $\Delta F_1 = 0$ transitions that an increase by a factor of four or five in the number of excited state molecules entering the cavity would be sufficient to start maser oscillation at this frequency. This improvement in performance would seem to be attainable with careful design of the maser cavity, source and focuser.

Since the absolute frequency of the superimposed $\Delta F_1 = 0$ transitions is known to within one or two kHz from POSENER's work ([3]), the pertinent knowledge of the HDO parameters given by the present experiment can be derived from a measurement of the hyperfine separations relative to this line. The sharpness and strength of this central transition therefore suggested a method of frequency measurement originally used on the quadrupole satellites in NH_3 ([5,6]), involving modulation of the klystron repeller voltage to produce sideband images of the main transition. Using these sidebands as frequency standard markers, the separation of the hyperfine components has been measured to within one kHz (Table I).

TABLE I.

Hyperfine transition	High frequency separation (kHz)	Low frequency separation (kHz)
A	20.7 ± .5	—
B	77.56 ± .4	76.27 ± .9
C	106.86 ± .5	108.90 ± .7
D	164.97 ± .7	163.81 ± .9

Measured separations of the hyperfine lines from the main $\Delta F_1 = 0$ transitions. The errors quoted are probable errors.

The frequencies and intensities of the observed hyperfine components have been interpreted in terms of the Hamiltonian:

$$H = \frac{1}{2}\frac{(eq_J Q)_D}{I_D(2I_D-1)J(2J-1)}\left[3(\boldsymbol{I}_D\cdot\boldsymbol{J})^2+\frac{3}{2}(\boldsymbol{I}_D\cdot\boldsymbol{J})-I_D^2J^2\right]+$$

$$+\,g_H g_D \mu_N^2\left[\frac{\boldsymbol{I}_H\cdot\boldsymbol{I}_D}{r^3}-\frac{3(\boldsymbol{I}_H\cdot\boldsymbol{r})(\boldsymbol{I}_D\cdot\boldsymbol{r})}{r^5}\right]+C_H\boldsymbol{I}_H\cdot\boldsymbol{J}+C_D\boldsymbol{I}_D\cdot\boldsymbol{J}\,.$$

Since the electric quadrupole coupling of the deuteron to the molecular rotation is the strongest interaction expected, a representation $F_1 F$ is appropriate, where

$$\boldsymbol{I}_D+\boldsymbol{J}=\boldsymbol{F}_1$$

$$\boldsymbol{I}_H+\boldsymbol{F}_1=\boldsymbol{F}\,.$$

([5]) B. P. DAILEY, R. L. KYHL, M. W. P. STRANDBERG, J. H. VAN VLECK and E. B. WILSON Jr.: *Phys. Rev.*, **70**, 984 (1946).

([6]) J. W. SIMMONS and W. GORDY: *Phys. Rev.*, **73**, 713 (1948).

The matrix elements of the Hamiltonian in this representation may be found in the standard works on atomic and molecular spectroscopy; POSENER (7) tabulates them in a particularly concise form for a molecule containing several nuclear spins.

In an asymmetric rotor the coupling constants of each of the terms in the above Hamiltonian can in general be expected to vary from rotational state to state. In the present case, however, the close symmetry of the observed hyperfine structure about the main line, and the unresolved appearance of this transition itself, indicate that these constants vary at most by only a few kHz between the 2_{20} and 2_{21} levels.

Only three constants therefore have been required to fit the observed spectrum, the constant for the spin-spin term being calculated from the known molecular geometry ($r_{OH} = .9584$ Å $\sphericalangle$ HOD $= 104° 27'$) (8) and nuclear magnetic moments. A fit to within 0.5 kHz of the *average* of the high and low frequency components has been obtained with:

$$(eq_z Q)_D = (78.3 \pm 0.6) \text{ kHz},$$

$$C_D = (\ 2.0 \pm 0.2) \text{ kHz},$$

$$C_H = (43.3 \pm 0.4) \text{ kHz}.$$

The calculated hyperfine structure is shown in Fig. 2. The weak $2\frac{5}{2} - 3\frac{5}{2}$ transition near 184 kHz has been observed on the high frequency side, but was not measured.

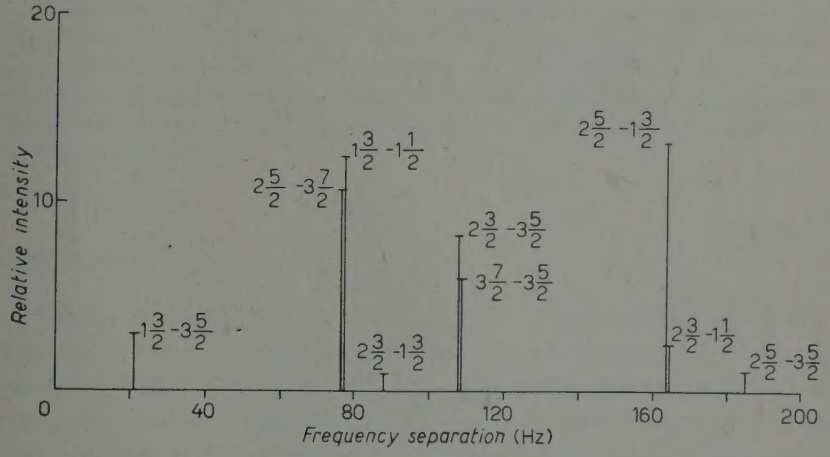

Fig. 2. – Hyperfine structure calculated on the assumption of equal coupling parameters in the 2_{20} and 2_{21} states. The main $\Delta F_1 = 0$ transitions, which all lie at the origin of the above figure, have a total intensity of 321 on the scale shown.

The differences between upper and lower state constants and those given above are calculated to fall within the quoted errors.

(7) D. W. POSENER: *Austr. Journ. Phys.* **11**, 1 (1958).

(8) G. HERZBERG: *Infrared and Raman Spectra of Polyatomic Molecules* (New York, 1945), p. 489.

The assumption of cylindrical symmetry of the molecular electric fields about the OD bond predicts in the present case, quite by coincidence, an equality of the quadrupole coupling constants in the 2_{20} and 2_{21} states to about 0.1 percent. Making this assumption, $(eqQ)_{OD} = 312.5$ kHz. The limits given on q_J imply an upper limit of 0.15 for the magnitude of the asymmetry parameter of the field gradient tensor,

$$\eta = \left(\frac{\partial^2 V}{\partial x_m^2} - \frac{\partial^2 V}{\partial y_m^2}\right) \Big/ \frac{\partial^2 V}{\partial z_m^2},$$

where z_m is the principal axis of this tensor assumed to lie along the OD bond. For a slightly asymmetric rotor, the $I \cdot J$ coupling constants are not expected to vary much between two states which in the symmetric top limit go over to the same rotational level, except for the case where $|K| = 1$.

* * *

The authors are indebted to Professor C. H. TOWNES for his attention and suggestions during all stages of the present work. Helpful discussions with Dr. A. JAVAN and correspondence with Dr. D. POSENER are gratefully acknowledged.

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